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STOCHASTIC MODELLING OF EM SCATTERING FROM FOLIAGE

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During this project, we focused on the development and evaluation of analytical methods for the description of scattering and absorption of electromagnetic radiation by "dense" foliage. The key consideration has been the description of multiple scattering processes in a random medium. Three different techniques for the description of microscopic scattering processes are discussed in this report. The T-matrix approach to multiple scattering represents the field at an individual scattering center in terms of an "equivalent" field produced by the other scatterers. It supports several natural approximations for this equivalent field which take multiple scattering processes into account at various levels of detail. The Coherent Potential Approximation is developed in detail. The homogenization method is a technique for the derivation of an equivalent representation of the scattering process in terms of an asymptotic analysis of Maxwell's equations in the limit as the separation between the scattering centers approaches zero. Bounds for the effective parameter representations are also discussed. There has been some very interesting work in this area based (over)

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on the classical work of Hashin and Shtrikman. We summarize aspects of this work that relate to the problem of scattering in foliage. Scattering from foliage covered terrain on a macroscopic scale is treated by using the approximations to develop effective media representations of the foliage. These are used in combination with an analytical methodology based on the method of smoothing perturbation to treat the foliage as an "interface" region with a "very rough" boundary.

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1 Introduction and Summary

This project has been concerned with basic research on analytical methods for representation of multiple scattering effects in heterogeneous random media. We have examined several methods for representation and computation of effective parameter models for the media in terms of statistical characterizations of the media. These methods lead to representation for the effective parameters, e.g., the effective dielectric constant or conductivity in a random medium. Since these representations may be difficult to evaluate in specific cases, we have also examined current research on the derivation of *bounds* for the effective parameters as a means of quickly providing good approximations to the behavior of the medium. Finally, we have included a computational technique for the evaluation of simplified scattering (and absorption) representations in terms of transmission and reflection coefficients for EM radiation encountering a region of foliage.

Our primary objective has been analysis of the interaction of EM radiation with microscopic scattering elements, retaining multiple scattering phenomena in the final analytical representation of the physics of the interaction. Since the geometries assumed by foliage as a scattering media are random and may include several different types of elementary scatterers, the physics is extremely complex. Figure 1 shows the scattering cross section of a single tree in the frequency range of interest here. It is apparent from the figure that the infrastructure of the tree plays a significant role in determining the radar cross section.

We have focussed our research on analytical tools capable of producing systematic,

computationally feasible approximations to the multiple scattering process. We have concentrated our efforts on the evaluation of *effective parameter* representations of the scattering medium which retain multiple scattering processes in the definition and computation of the effective parameter models. We have examined three different methods for this purpose:

1. Methods based on the *Transition Matrix* formalism which represent the explicit interaction of elementary scatterers and permit the systematic construction of approximations.

2. Bounds for the effective paramters (dielectric constant) representing the medium based on the generation of certain "trial fields" and the use of geometry dependent and geometry independent formulations.

3. Homogenization of the medium via an asymptotic analysis of Maxwell's equations in differential form producing an approximation to the total fields in the heterogeneous medium in terms of the corresponding fields in a medium with a homogeneous model (e.g., dielectric constant).

Our work on the T-matrix methods is summarized in Section 2. It is important to emphasize that the T-matrix methods, though formal in structure, provide simple models for multiple scattering phenomena which are valid for a wide range of frequencies (of the incident radiation). We have described a series of approximations (ATA, CPA, EMA, SCA) based on the T-matrix formalism. We have also shown how these approximations can be developed systematically by associating the T-matrix formalism

with a class of variational problems originally used to derive bounds for the effective parameters of composite materials.

In Section 3 we continue the examination of the derivation and use of "bounds" for the effective parameters in a scattering problem. The derivation of bounds is an old subject in scattering theory, dating from early in this century. Nevertheless, there has been some very interesting recent work on various methods for deriving bounds. We discuss this work briefly in Section 3, as it applies to the problem of scattering from foliage. As with homogenization, the class of bounds discussed applies mostly to the low frequency case.

The work on the use of "multiple-scale" asymptotic analysis (homogenization) for the representation of scattering processes is given in Section 4. This is primarily a "low frequency" theory, valid when the wave length of the incident radiation exceeds the characteristic dimensions of inter-particle separation in the scattering medium. The homogenization method does, however, provide a systematic procedure for the construction of high order approximations to the "effective parameter" representations of the scattering process. These approximations may be evaluated by common numerical procedures when the scattering medium has a periodic structure. We describe this case first, before turning to the more realistic, but less tractable, case when the physical model parameters (permittivity, conductivity, and permeability) are random functions of position in the medium.

In Section 5 we describe a procedure combining an analytical representation for the macroscopic scattering phenomena from a "layer" of foliage above terrain with the effec-

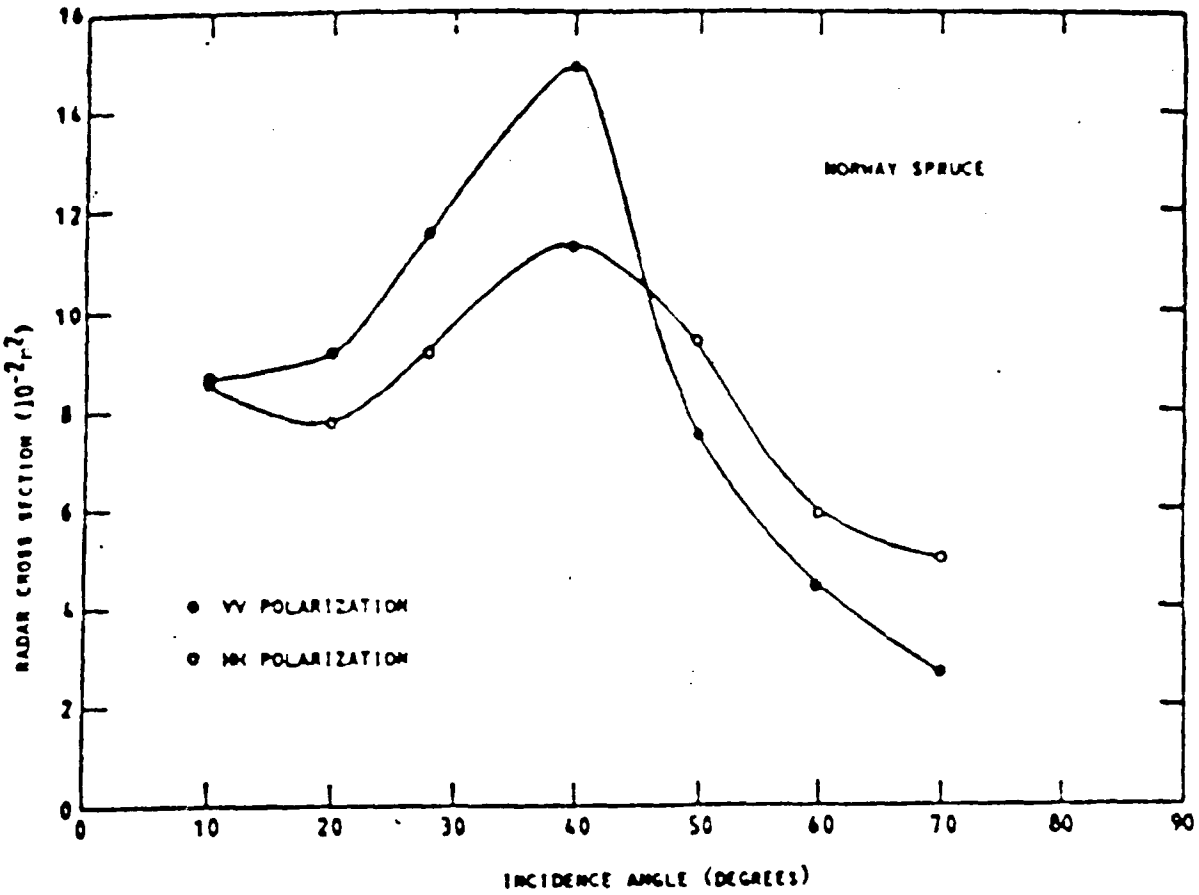

Figure 1.1: Angular variation of the synthetic pulse response from a spruce tree at 8 to 12 GHz

tive parameter representations of the microscopic interactions of the EM radiation and the elementary scatterers (absorbing centers) in the foliage. The method of smoothing perturbation is used to obtain a deterministic vector wave equation for the average electric field scattered by the random "very rough" surface. The coherent scattering is characterized by reflection coefficients which can be computed easily. The technique is applied to treat surfaces with uniform and Gaussian distributed height variations, and the results are compared to the Fresnel coefficients. Only small differences are observed at grazing incidence; however, at larger incident angles substantial differences (20-50%) in both amplitude and phase variations are obtained.

In Section 6 we present some conclusions and recommendations for further work.

2 T-matrix Approximations to Multiple Scattering

In this section we use the transition matrix (T-matrix) formalism to describe multiple scattering phenomena in a region containing two or more types of dielectric scatterers arranged in a random geometry in a surrounding background medium. Models for scattering from a medium containing a two (or more) classes of (randomly oriented) scatterers in an enveloping medium are prescribed in terms of the definition and derivation of approximations for the "effective dielectric constant" resulting in a family of approximations for an "effective scattering representation" for heterogenous media.

2.1 Basic Framework

Consider a region $\mathcal{O}$ of space occupied by a homogeneous (background) dielectric medium with dielectric constant ϵ_0 containing elements from m classes of scatterers with dielectric constants $\epsilon_1, \ldots, \epsilon_m$. Let $v_{ij}, i = 1, \ldots, m, j = 1, 2, \ldots, N_i$, be the subsets of $\mathcal{O}$ occupied by scattering elements j of class i. The *dielectric properties* of the composite material in $\mathcal{O}$ are described by

$$\epsilon(r) = \epsilon_0 + \sum_{i=1}^{m} (\epsilon_i - \epsilon_0) \sum_{j=1}^{N_i} \chi_{v_{ij}}(r) \tag{2.1}$$

where

$$\chi_{v_{ij}}(r) = \begin{cases} 1 & r \in v_{ij} \\ 0 & r \notin v_{ij} \end{cases} \tag{2.2}$$

(See Figures 2.1,2.2.) Let δ_i be a (dimensionless) parameter describing a characteristic dimension of elementary scatterers in class i (e.g., the radius of spherical scatterers);

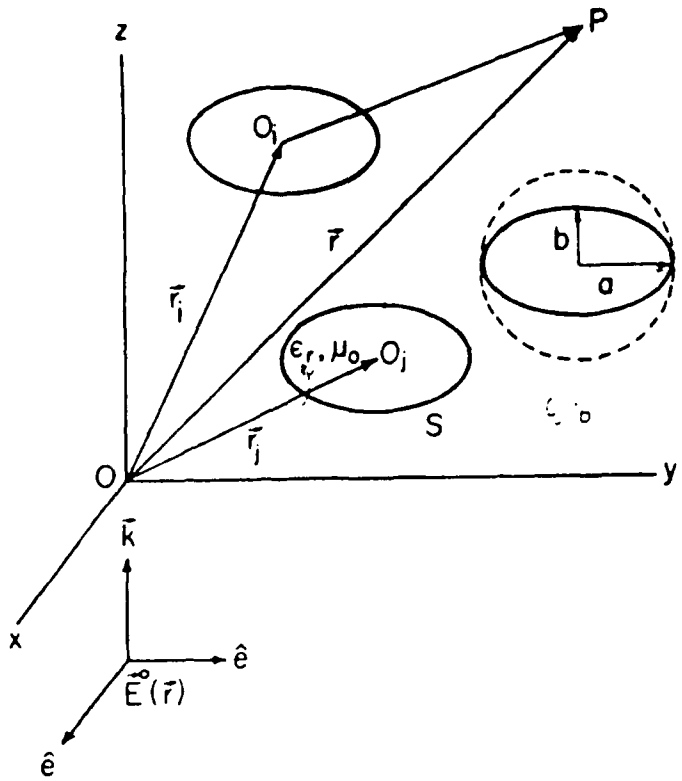

Figure 2.1: Distribution of scatterers and plane wave incidence.

and let ρ_i be the total volume fraction of $\mathcal{O}$ occupied by scatterers of class $i = 1, \ldots, m$. Suppose a constant (in amplitude - time harmonic) field is incident on the region $\mathcal{O}$.

We wish to characterize the scattering properties of the composite material in the limit $N_i \to \infty$, $\delta_i \to 0$, with $\rho_i, , i = 1, \ldots, m$ constant. We are particularly interested in evaluating the relative roles of the classes of scatterers in the scattering process, e.g., in terms of their relative densities ρ_i in the cases (a) $\rho_i, \rho_j \to 0$ with $\rho_i/\rho_j \sim O(1)$ and (b) $\rho_i, \rho_j \to 0$ with $\rho_i/\rho_j \to 0$.

We shall sketch the development of a family of effective scattering approximations for the material using the multiple scattering framework developed originally by M. Lax (1951,1952).

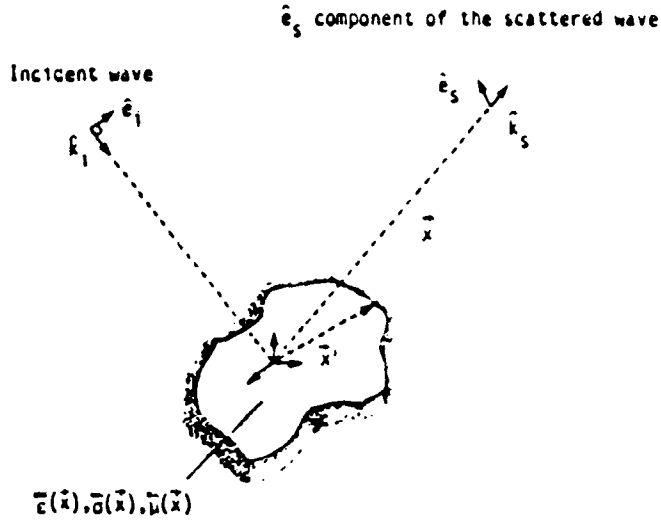

Figure 2.2: Plane wave scattering by an arbitrary inhomogeneous, anisotropic object.

2.2 Transition Operator Representations

We use the *transition operator* formalism (Lax 1951,1952,1973) as used by (Lang 1981) and (Kohler and Papanicolaou 1981) among many others, as a framework for the derivation of several formulas for an *effective dielectric constant* for the composite medium. Maxwell's equations for this situation may be written in the form

$$\nabla \cdot (\epsilon_0 E) + \sum_{i=1}^{m} (\epsilon_i - \epsilon_0) \sum_{j=1}^{N_i} \nabla \cdot [\chi_{v_{i_j}} E] = 0 \tag{2.3}$$

$$\nabla \times E = 0$$

with the boundary condition

$$< E > \rightarrow \overline{E} \qquad \text{as} \quad N_i \rightarrow \infty \tag{2.4}$$

where $< \cdot >$ is expectation and $\overline{E}$ is the constant external field. The constituent relationship

$$D(x) = \epsilon(x)E(x) \tag{2.5}$$

and the condition

$$< D(\cdot) >= \epsilon^* \overline{E} \tag{2.6}$$

defines the *effective dielectric constant* for the composite material.

Rewriting (2.3) in abstract form

$$(L_0 + M)E = 0, \qquad \nabla \times E = 0 \tag{2.7}$$

where

$$\begin{gathered} L_0 = \nabla \cdot (\epsilon_0 \cdot) \\ M = M_1 + \cdots + M_m, \qquad M_i = \sum_{j=1}^{N_i} V_{ij} \\ V_{ij} = (\epsilon_i - \epsilon_0) \cdot \nabla[\chi_{v_{ij}}(\cdot)] \end{gathered} \tag{2.8}$$

we have

$$E + L_0^{-1} M E = F \tag{2.9}$$

as the integral form of (2.3). F is chosen so that

$$L_0 F = 0, \qquad < E >= \overline{E}. \tag{2.10}$$

The *transition operator* ("matrix") T is defined as follows:

$$L_0^{-1} M = (L_0 + M - M)^{-1} M$$

$$= [(L_0 + M)(I - (L_0 + M)^{-1}M]^{-1}M \tag{2.11}$$

$$= T(I - T)^{-1}$$

with $T = (L_0 + M)^{-1}M$. So (2.9) becomes

$$[I + T(I - T)^{-1}]E = F \tag{2.12}$$

or

$$E = (I - T)F \tag{2.13}$$

The condition (2.10) gives

$$F = (I - < T >)^{-1}\overline{E} \tag{2.14}$$

$$E = (I - T)(I - < T >)^{-1}\overline{E} \tag{2.15}$$

Using (2.5)(2.6)(2.15), we have

$$\epsilon^*\overline{E} = (< \epsilon > - < \epsilon T >)(I - < T >)^{-1}\overline{E} \tag{2.16}$$

as the definition of ϵ^* in terms of the transition operator.

It is apparent that knowledge of T completely characterizes the scattering properties of the composite medium. The operator T is related to the scattering amplitudes of the individual particles, and to their interactions. For simple (single) dipole scatterers, T can be determined from the polarizability of the element; and so, T is defined in terms of quantities of physical interest. Other properties of the transition operator are described in (Lax 1951,1952,1973), (Lang 1981), and (Kohler and Papanicolaou 1981). Our analysis of the representation (2.15)(2.16) is based on (Kohler and Papanicolaou 1981).

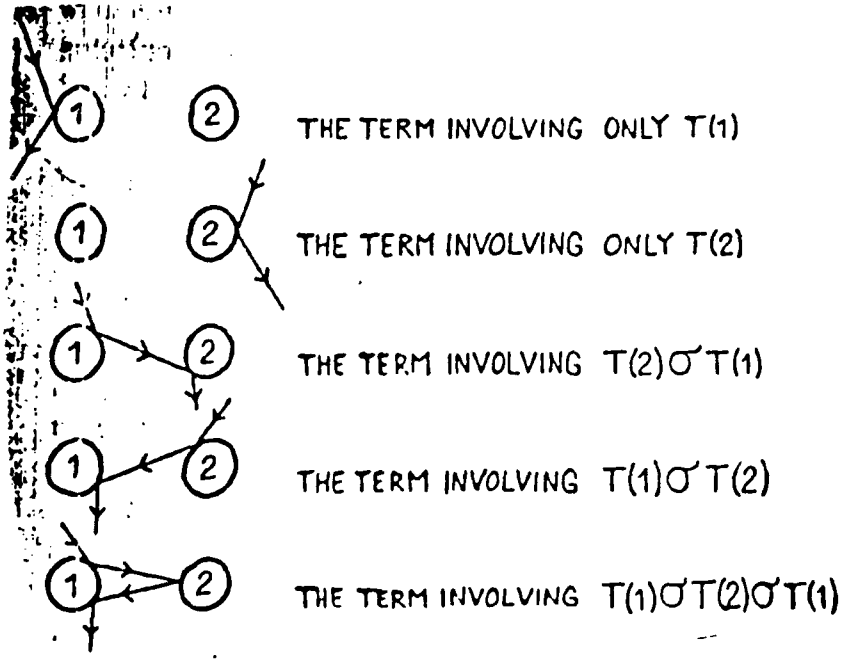

Figure 2.3: Multiple scattering interpretation of the T-matrix formalism for two scatterers.

To bring out the interaction of the various classes of scatterers, (see Figure 2.2 we reconsider (2.9)

$$E + L_0^{-1}(M_1 + \cdots + M_m)E = F \tag{2.17}$$

which may be written as

$$E = F - T_1E_1 - T_2E_2 - \cdots T_mE_m \tag{2.18}$$

with

$$\begin{aligned} L_0^{-1}M_i &= T_i(I - T_i)^{-1} \\ T_i &= (L_0 + M_i)^{-1}M_i \\ E_i &= (I - T_i)^{-1}E \end{aligned} \tag{2.19}$$

since $E_i - T_iE_i = E$, we have

$$E_i + \sum_{k \neq i}^{m} T_kE_k = F \tag{2.20}$$

Equations (2.18)-(2.20) express the field in terms of the fields and transition operators associated with each *class* of scatterers in the composite medium.

For $m = 2$ classes of scatterers we have

$$\begin{aligned} T &= (L_0 + M_1 + M_2)^{-1}(M_1 + M_2) \\ &= (I + T_{12})^{-1}T_1 + (I + T_{21})^{-1}T_2 \end{aligned} \tag{2.21}$$

where the pair transition operators are

$$T_{ij} = (L_0 + M_i)^{-1}M_j, \qquad i \neq j = 1, 2, \ldots, m \tag{2.22}$$

This representation gives the formula

$$\epsilon^* \sim [< \epsilon > - < \epsilon(I + T_{12})^{-1}T_1 > - < \epsilon(I + T_{21})^{-1}T_2 >] \tag{2.23}$$

$$\cdot[I - < (I + T_{12})^{-1} T_1 > - < (I + T_{21})^{-1} T_2 >]^{-1}$$

for the effective dielectric constant in which the roles of each class of scatterers and their interaction is explicit. We could continue this process to make the role of each elementary scatterer of class i explicit. Note that (2.23) contains no approximations.

For the general case of $m \geq 2$ classes of embedded scatterers the formula is

$$\epsilon^* \sim \left[< \epsilon > - \sum_{i=1}^{m} \sum_{j \neq i} < \epsilon (I + T_{ij})^{-1} T_i > \right] \left[1 - \sum_{i=1}^{m} \sum_{j \neq i} < (I + T_{ij})^{-1} T_i > \right]^{-1} \tag{2.24}$$

2.3 Average T-Matrix Approximation (ATA)

Consider the case $m = 2$ for a moment. Expanding the expression for the T-operator, we have

$$T = T_1 + T_2 - T_{12} T_1 - T_{21} T_2 + \cdots \tag{2.25}$$

neglecting the higher order terms ($T_{12} T_1, \ldots$), and substituting in (2.23), we obtain the simple approximation

$$\epsilon^* \sim [< \epsilon > - \sum_{i=1}^{2} < \epsilon T_i >][I - \sum_{i=1}^{2} < T_i >]^{-1} \tag{2.26}$$

called the *average T-matrix approximation* (ATA) (Lax 1951,1973) (Kohler and Papanicolaou 1981). The expressions $< T_j >$ and $< \epsilon T_j >$ in (2.26) diverge in typical cases (Batchelor 1974) (Gubernatis 1978), so the ATA must be used with care. The usual technique is to employ a "cutoff" argument, limiting attention to a finite, but large region of space and ignoring contributions from beyond this region.

It is possible to evaluate (2.26) in the limit as the sizes of the elementary scatterers approaches zero. Suppose each class consists of spherical scatterers having radii $\delta_i, i = 1,2$ with

$$\rho_i = \frac{4}{3}\pi\delta_i^3 c_i \tag{2.27}$$

the volume fraction occupied by each class. Here c_i is the average number of sphere centers per unit volume. this case leads to the approximation

$$\epsilon^* \sim \frac{< \epsilon > + \rho_1\epsilon_1\left[\frac{\epsilon_0-\epsilon_1}{2\epsilon_0+\epsilon_1}\right] + \rho_2\epsilon_2\left[\frac{\epsilon_0-\epsilon_2}{2\epsilon_0+\epsilon_2}\right]}{1 + \rho_1\left[\frac{\epsilon_0-\epsilon_1}{2\epsilon_0+\epsilon_1}\right] + \rho_2\left[\frac{\epsilon_0-\epsilon_2}{2\epsilon_0+\epsilon_2}\right]} \tag{2.28}$$

which is a version of the *Claussius - Mossotti* formula (Kohler and Papanicolaou 1981, p. 213) (Landauer 1978). In general, the Claussius - Mossotti formula is viewed as a good approximation for low to moderate volume fractions ρ_i.

The ATA not only contains divergent terms as a rule, it also fails to account for the interactions between scatterers. A family of more refined approximations has been developed to include interaction effects in formulas for effective scattering representations which are nonetheless computationally feasible (Lax 1951,1952,1973) (Elliott et al 1974).

2.4 Coherent Potential Approximation (CPA)

Among the most interesting of these is the *Coherent Potential Approximation* (CPA) which essentially involves neglecting the difference between the field exciting the medium and the average field. Let ϵ_r be a "reference" dielectric constant to be selected, and

consider Maxwell's equations rewritten as

$$\nabla \cdot [\epsilon_r E] + \sum_{i=1}^{m} \sum_{j=1}^{N_i} \nabla \cdot [(\epsilon_i - \epsilon_0)\chi_{ij} E]$$

$$+ \sum_{i=1}^{m} \sum_{j=1}^{N_i} \left(\frac{\epsilon_0 - \epsilon_r}{m N_i} \right) \nabla \cdot E = 0 \tag{2.29}$$

$$\nabla \times E = 0$$

We may write this abstractly as

$$(L_r + M^r)E = 0 \tag{2.30}$$

with

$$L_r = \nabla \cdot (\epsilon_r \cdot)$$

$$M^r = \sum_{i=1}^{2} \sum_{j=1}^{N_i} V_{ij}^r \tag{2.31}$$

$$V_{ij}^r = \nabla \cdot [(\epsilon_i - \epsilon_0)\chi_{ij} \cdot] + \left(\frac{\epsilon_r - \epsilon_0}{m N_i} \right) \nabla \cdot$$

The same argument as before yields the representation

$$\epsilon^* \sim [< \epsilon > - \sum_{i=1}^{m} \sum_{j=1}^{N_i} < \epsilon T_{ij}^r >][1 - \sum_{i=1}^{m} \sum_{j=1}^{N_i} < T_{ij}^r >]^{-1} \tag{2.32}$$

where

$$T_{ij}^r = [L_r + V_{ij}^r]^{-1} V_{ij}^r \tag{2.33}$$

The CPA is based on choosing ϵ_r to "optimize" the approximation in (2.32).

Let $g(x)$ be a smooth function and consider the evaluation of T_{ij}^r

$$w^{ij}(x) = (T_{ij}^r g)(x) \tag{2.34}$$

so

$$\nabla \cdot [\epsilon_r w^{ij}] + \nabla \cdot [(\epsilon_i - \epsilon_0)\chi_{v_{ij}} w^{ij} + \left(\frac{\epsilon_0 - \epsilon_r}{mN_i}\right) w^{ij}]$$

$$= \nabla \cdot [(\epsilon_i - \epsilon_0)\chi_{v_{ij}} g + \left(\frac{\epsilon_0 - \epsilon_r}{mN_i}\right) g] \tag{2.35}$$

$$\nabla \times w^{ij} = 0$$

Let

$$\epsilon_i^{r,N} = [\epsilon_r + \frac{1}{mN_i}(\epsilon_0 - \epsilon_r)] \approx \epsilon_r \quad \text{for} \quad N_i \text{ large} \tag{2.36}$$

and $w^{ij} = w_1^{ij} + w_2^{ij}$ where

$$\nabla \cdot (\epsilon_i^{r,N} w_1^{ij}) + \nabla \cdot [(\epsilon_i - \epsilon_0)\chi_{v_{ij}} w_1^{ij}] \tag{2.37}$$

$$= \nabla \cdot [(\epsilon_i - \epsilon_0)\chi_{v_{ij}} g]$$

$$\nabla \cdot (\epsilon_i^{r,N} w_2^{ij}) + \nabla \cdot [(\epsilon_i - \epsilon_0)\chi_{v_{ij}} w_2^{ij}] \tag{2.38}$$

$$= \left(\frac{\epsilon_0 - \epsilon_r}{mN_i}\right) \nabla \cdot g$$

$$\nabla \times w_1^{ij} = 0 = \nabla \times w_2^{ij}$$

Let $\{y_{ij}^N\}$ be the locations of the centers of the individual scattering elements. For δ_i small, we may replace g on the right in (2.37) by $g(y_{ij}^N) = g_{ij}$. Using this and the approximation

$$\epsilon_i^{r,N} \approx \epsilon_r \quad \text{for} \quad N_i \text{ large} \tag{2.39}$$

we have

$$w_1^{ij}(x) \sim -\frac{\epsilon_0 - \epsilon_i}{3\epsilon_r + \epsilon_i - \epsilon_0} \delta_i^3 \nabla_x \left[\frac{g_{ij} \cdot (x - y_{ij}^N)}{|x - y_{ij}^N|^3}\right], \quad |x - y_{ij}^N| > \delta_i \tag{2.40}$$

$$w_1^{ij}(x) \sim -\frac{\epsilon_0 - \epsilon_i}{3\epsilon_r + \epsilon_i - \epsilon_0} g_{ij}, \quad |x - y_{ij}^N| < \delta_i$$

a result similar to (Kohler and Papanicolaou 1981, sect. 8). In addition, using (2.39) in (2.38)

$$w_2^{ij} = \frac{1}{4\pi m N_i} \nabla_x \int \frac{(x-y)\cdot g(y)}{|x-y|^3} dy + w_3^{ij} \tag{2.41}$$

where

$$w_3^{ij}(x) \sim + \frac{\epsilon_0 - \epsilon_i}{3\epsilon_r + \epsilon_i - \epsilon_0} \frac{1}{m N_i} \delta_i^3 \nabla_x \left[\frac{g_{ij}\cdot(x - y_{ij}^N)}{|x - y_{ij}^N|^3} \right], \quad |x - y_{ij}^N| > \delta_i \tag{2.42}$$

$$w_3^{ij}(x) \sim + \frac{\epsilon_0 - \epsilon_i}{3\epsilon_r + \epsilon_i - \epsilon_0} \frac{1}{m N_i} g_{ij}, \quad |x - y_{ij}^N| < \delta_i$$

Combining these, we have

$$\sum_{j=1}^{N_i} (T_{ij}^r g)(x) \sim \frac{1}{4\pi m} \nabla_x \int \frac{(x-y)\cdot g(y)}{|x-y|^3} dy$$

$$\frac{\epsilon_0 - \epsilon_i}{3\epsilon_r + \epsilon_i - \epsilon_0} \delta_i^3 \sum_{j=1}^{N_i} (1 - \chi_{v_{ij}}(x)) \nabla_x \left[\frac{g_{ij}\cdot(x - y_{ij}^N)}{|x - y_{ij}^N|^3} \right] \tag{2.43}$$

$$\frac{\epsilon_0 - \epsilon_i}{3\epsilon_r + \epsilon_i - \epsilon_0} \sum_{j=1}^{N_i} \chi_{v_{ij}}(x) g_{ij} + \sum_{j=1}^{N_i} w_3^{ij}(x)$$

Arguing as in (Kohler and Papanicolaou 1981, sect. 8), the last term on the right in (2.43) is $O(N_i^{-1})$ on the average as $N_i \to \infty$. We take the expectation and choose ϵ_r so that

$$< \sum_{i=1}^{m} \sum_{j=1}^{N_i} T_{ij}^r > = 0 \tag{2.44}$$

This is the CPA. This requires

$$\frac{\epsilon_0 - \epsilon_r}{3\epsilon_r} = \sum_{i=1}^{m} \rho_i \frac{\epsilon_i - \epsilon_0}{3\epsilon_r + \epsilon_i - \epsilon_0} \tag{2.45}$$

where $\rho_i = \frac{4}{3}\pi\delta_i^3 c_i$ with c_i the average number of scattering centers (of class i) per unit volume.

Using this choice of ϵ_r, the effective dielectric constant may be approximated using the CPA by evaluating (2.32). The result is

$$\epsilon^* = \epsilon_0 + \left(\sum_{i=1}^{m} \rho_i \epsilon_i - \rho\epsilon_0\right) - (1-\rho)\epsilon_0 \left(\frac{\epsilon_0 - \epsilon_r}{3\epsilon_r}\right) \tag{2.46}$$

$$+ \left(\sum_{i=1}^{m} \rho_i \epsilon_i \frac{\epsilon_0 - \epsilon_i}{3\epsilon_r + \epsilon_i - \epsilon_0}\right) - \left(\sum_{i=1}^{m} \rho_i \epsilon_i\right) \left(\frac{\epsilon_0 - \epsilon_i}{3\epsilon_r + \epsilon_i - \epsilon_0}\right)$$

where

$$\rho = \sum_{i=1}^{m} \rho_i$$

This result should be contrasted with the formula from *Effective Medium Theory (EMA)* (Elliot et al 1974) which gives

$$\frac{\epsilon^* - \epsilon_0}{2\epsilon^* - \epsilon_0}(1-\rho) + \sum_{i=1}^{m} \rho_i \left(\frac{\epsilon^* - \epsilon_i}{2\epsilon^* - \epsilon_i}\right) = 0 \tag{2.47}$$

The most accurate formula is obtained by using ϵ^* as the dielectric constant of the "reference medium." This leads to the *Self-Consistent (SCA)* formula

$$\epsilon^* = \frac{< \epsilon > + \sum_{i=1}^{m} \rho_i \epsilon_i \left(\frac{\epsilon_0 - \epsilon_i}{3\epsilon^* + \epsilon_i - \epsilon_0}\right)}{1 + \sum_{i=1}^{m} \rho_i \left(\frac{\epsilon_0 - \epsilon_i}{3\epsilon^* + \epsilon_i - \epsilon_0}\right)} \tag{2.48}$$

The results from several approximations (loss versus volume fraction) are compared in Figure 2.4. It is clear that there is considerable variation among the methods.

2.5 Selection of Trial Fields Based on Variational Principles

The results from the previous subsection can be put in perspective by considering the problem of constructing bounds for the effective parameters representing the medium.

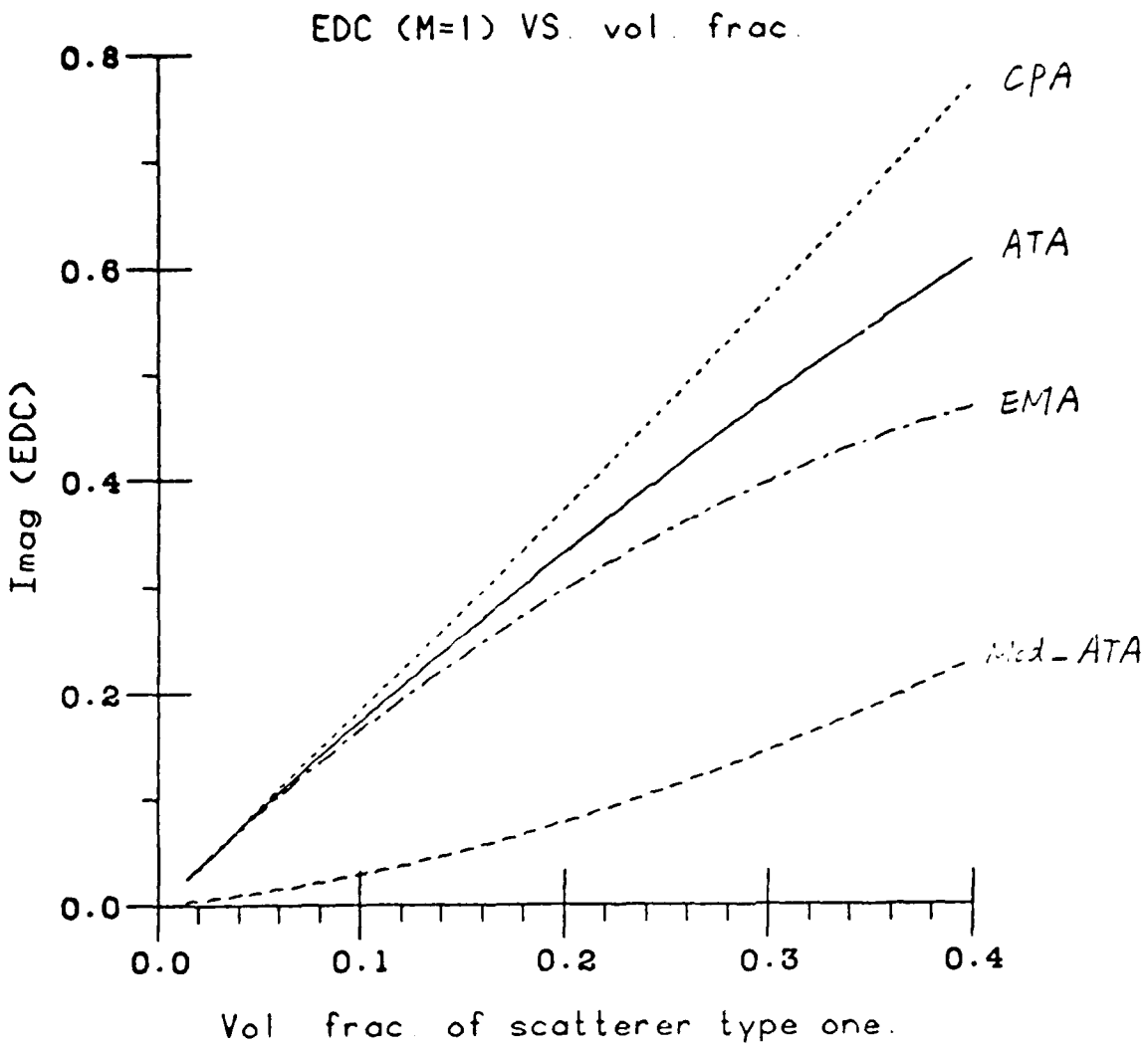

Figure 2.4: Comparison of the effective parameter for several approximation methods.

In particular, we can develop the process of selecting a "reference background medium" in the context of a variational problem.

Let $\mathcal{E}$ be a curl free field, and

$$V = \frac{1}{2} < \epsilon\mathcal{E}, \mathcal{E} >, \quad \nabla \times \mathcal{E} = 0, \quad < \mathcal{E} >= \bar{E} \tag{2.49}$$

where $V(\cdot)$ is the electrostatic energy in $\mathcal{E}$. Then $\min_{\mathcal{E}} V(\mathcal{E})$ is achieved by E satisfying Maxwell's Equations

$$\begin{gathered} D(x) = \epsilon(x)E(x) \\ \nabla \times E = 0, \quad \nabla \cdot D = 0 \\ < E >= \bar{E} \end{gathered} \tag{2.50}$$

where $\bar{E}$ is a given external field. Recall $< D >= \epsilon^* \bar{E}$ defines the effective dielectric constant. If $\tilde{\mathcal{E}}$ is any field with $\nabla \times \tilde{\mathcal{E}} = 0, < \tilde{\mathcal{E}} >= 0$, then

$$\frac{1}{2} < \epsilon E \cdot \tilde{\mathcal{E}} >= 0 \tag{2.51}$$

Therefore,

$$\begin{aligned} V(E) &= \frac{1}{2} < \epsilon E \cdot E > \\ &= \frac{1}{2} < \epsilon E \cdot \bar{E} > + \frac{1}{2} < \epsilon E \cdot (E - \bar{E}) > \\ &= \frac{1}{2} < \epsilon E > \cdot \bar{E} \\ &= \frac{1}{2} < \epsilon > \bar{E} \cdot \bar{E} \end{aligned} \tag{2.52}$$

This gives the general upper bound

$$\epsilon^* \bar{E} \cdot \bar{E} \leq \quad < \epsilon\mathcal{E} \cdot \mathcal{E} > \tag{2.53}$$

for any stationary random field $\mathcal{E}$ with $\nabla \times \mathcal{E} = 0, < \mathcal{E} >= \bar{E}$. Inequality (2.53) gives an upper bound on ϵ^* for any "trial field" $\mathcal{E}$.

To obtain a lower bound, we use the complementary variational principle. Let

$$W(\mathcal{D}) = \frac{1}{2} < \epsilon^{-1}\mathcal{D} \cdot \mathcal{D} > \tag{2.54}$$

$$\nabla \cdot \mathcal{D} = 0, \quad < \mathcal{D} >= \bar{D}$$

The minimum is again achieved by Maxwell's equations. This provides the bound

$$(\epsilon^*)^{-1}\bar{D} \cdot \bar{D} \leq \quad < \epsilon^{-1}\mathcal{D} \cdot \mathcal{D} > \tag{2.55}$$

for any trial field $\mathcal{D}$ with $\nabla \cdot \mathcal{D} = 0, < \mathcal{D} >= \bar{D}$.

To construct suitable trial fields, we use the transition operator – multiple scattering formalism. Recall the formulation of Maxwell's equations for the multicomponent medium

$$\nabla \cdot (\epsilon_0 E) + \sum_{i=1}^{m} \sum_{j=1}^{N_i} \nabla \cdot \left[(\epsilon_i - \epsilon_0) \chi_{v_{ij}} E \right] = 0 \tag{2.56}$$

$$\nabla \times E = 0$$

which we write in abstract form as

$$(L_0 + M)E = 0, \quad \nabla \times E = 0 \tag{2.57}$$

As before

$$E + L_0^{-1} M E = F \tag{2.58}$$

where F satisfies $L_0 F = 0, < E >= \bar{E}$. In terms of the transition operator $T = (L_0 + M)^{-1} M$ we may write this as

$$E = (I - T)F$$

$$= (I - \sum_{i=1}^{m} T_i)F \tag{2.59}$$

$$= (I - \sum_{i=1}^{m}\sum_{j=1}^{N_i} T_{ij})F$$

Based on this we define the trial field

$$\mathcal{E} = F - \sum_{i=1}^{m}\sum_{j=1}^{N_i} T_{ij}F \tag{2.60}$$

with F a constant to be chosen. Writing out the operator T_{ij} explicitly we have

$$\mathcal{E} = F - \sum_{i=1}^{m}\left(\frac{\epsilon_i - \epsilon_0}{2\epsilon_i + \epsilon_0}\right)\sum_{j=1}^{N_i}\left\{\left(1 - \chi_{v_{ij}}(x)\right)\delta^3\nabla\left[\frac{(x - y_{ij})\cdot F}{|x - y_{ij}|^3}\right] + \chi_{v_{ij}}(x)F\right\} \tag{2.61}$$

This trial field is curl free. We choose F so that $< \mathcal{E} >= \bar{E}$.

We shall continue this analysis in the next section which discusses general procedures for constructing bounds.

3 Bounds for Effective Parameters

In the previous section and in the next we present methods for representation and computation of effective parameter models for heterogeneous media in terms of statistical characterizations of the media. These methods lead to representations for the effective parameters, e.g., the effective dielectric constant and conductivity, which may be difficult to evaluate. In this section we continue the examination of the derivation of *bounds* on the effective parameters as a means of quickly providing good approximations to the behavior of the medium. We elaborate on the ideas introduced at the end of the previous section for construction of bounds based on variational procedures. We also consider the construction of bounds based on the method of analytical continuation introduced by Bergmann and extended by Willis, Golden and Papanicolaou.

Approximations to the effective permittivity (or conductivity in thermal systems) have been derived by three basic methods. The original work of Haskin and Shtrikman (1962) was based on variational principles, and these have been reformulated in recent years to provide one of the basic methods for deriving aproximations (Milton 1981a,b). The second method for deriving bounds is the method of "compensated compactness" (Tartar 1986). The third method involves the use of complex analysis and "representation formulas" (Bergmann 1978, Golden 1986, Golden and Papanicolaou 1983).

There has been a great deal of work in this area, not only in electromagnetics, but also in conductivity and elasticity. This work dates from the early years of this century (Bergmann 1978), and it includes some very recent work (e.g., (Ericksen 1986)).

We shall describe the applications of this work for the description of scattering and absorption of EM radiation by foliage covered terrain.

3.1 Definition of the Effective Parameters

The starting point is a definition of the effective parameter, e.g., the effective permittivity. We shall use a slightly more general model than the one used in the previous section. Suppose

$$\epsilon(x,\omega) = \epsilon_1\chi_1(\omega) + \cdots + \epsilon_m\chi_m(\omega) \tag{3.1}$$

is the dielectric "constant" of a composite medium, where $\epsilon_i, i = 1,\ldots,m$ is the (complex) dielectric constant of medium i, and the indicator function χ_i is 1 for all samples $\omega \in \Omega$ which have medium i at point x, and zero otherwise.[1] We use $(\Omega,\mathcal{F},\mathcal{P})$ to denote the probability space on which the statistics of the phenomena are defined. The quantity

$$\rho_j = \int_\Omega P(d\omega)\chi_j(\omega) \tag{3.2}$$

is the *volume fraction* occupied by the material of type $j = 1,2,\ldots,m$.

The spatially dependent random conductivity and magnetic permeability are similarly defined. In our case these functions depend on the random geometry of the scattering foliage which may be available only in terms of second order statistics. The bounds obtained for the "effective parameter" approximations of these functions should not depend on statistics beyond this level.

[1]Thus, we are assuming m classes of scatterers as in the previous section. We shall not have to deal with the individual scattering elements at this point.

If $E(x,\omega)$ and $D(x,\omega)$ are the electric and displacement field vectors (stationary in time), then

$$D_i(x,\omega) = \epsilon E_i(x,\omega) \tag{3.3}$$

$$\nabla \cdot D(x,\omega) = 0$$

$$\nabla \times E(x,\omega) = 0$$

and

$$\int_\Omega P(d\omega)E(x,\omega) = \bar{E}$$

provides the boundary condition, where $\bar{E}$ is a constant field incident on the region. By normalizing the field magnitudes and redefining the coordinates, we can take $\bar{E} = e_k$, the k^{th} unit vector in $\Re^3$. Let E^k and D^k denote the corresponding fields. Then (3.3) may be rewritten

$$D_i^k(x,\omega) = \sum_{ij=1}^{d} \epsilon_{ij}(x,\omega)E_j^k(x,\omega) \tag{3.4}$$

The *effective dielectric constant* may be defined as

$$\epsilon_{ik}^* = \int_\Omega P(d\omega)D_i^k(\omega) \tag{3.5}$$

In (Golden and Papanicolaou 1983) it was shown that this essemble average coincides with the usual definition involving a volume average.

Equation (3.5) may also be written in the symmetric form

$$\epsilon_{ik}^* = \int_\Omega P(d\omega) \sum_{j=1}^{d} \epsilon E_j^k \bar{E}_j^i \tag{3.6}$$

where the bar denotes complex conjugate. This expression suggests the interpretation of ϵ^* as the dielectric constant of a fictitious homogeneous medium (for the volume

V) which provides the same value for the electrostatic energy stored in the volume occupied by the heterogeneous medium.[2]

3.2 Variational Methods for Computing Bounds

In the last section we introduced the variational approach to the computation of bounds on the effective dielectric constant. In this subsection we shall develop this approach in detail – providing an assessment of the utility of the method for the treatment of scattering and absorption phenomonena in foliage and several formulas for effective parametric representations.

The starting point is the pair of variational principles:

$$\epsilon^* = \min_{\substack{\nabla \times \mathcal{E} = 0 \\ <\mathcal{E}> = e}} < \epsilon \mathcal{E} \cdot \mathcal{E} > \tag{3.7}$$

$$(\epsilon^*)^{-1} = \min_{\substack{\nabla \cdot \mathcal{D} = 0 \\ <\mathcal{D}> = e}} < \frac{1}{\epsilon} \mathcal{D} \cdot \mathcal{D} > \tag{3.8}$$

Here e is a unit vector (normalized external field), and $\mathcal{E}, \mathcal{D}$ run over stationary, square integrable random fields. The bracketts $< \cdot >$ as usual denote ensemble average.

The minima are attained by the solutions of Maxwell's equations (3.3). Note also that

$$\epsilon^* = < \mathcal{D} \cdot e > = < \epsilon \mathcal{E} \cdot e > = < \epsilon \mathcal{E} \cdot \mathcal{E} > \tag{3.9}$$

[2]The volume average form is $\epsilon^*_{ik} |E_0|^2 = \left[\int_V \epsilon(r) |E(r)|^2 dV \right] / meas(V)$ where V is the representative region, $meas(V)$ is its volume, and $E_0 = \left[\int_V E(r) dV \right] / meas(V)$ is the average (incident) field.

because $< \epsilon\mathcal{E} \cdot \tilde{\mathcal{E}} >= 0$ for any test field $\tilde{\mathcal{E}}$ with $< \tilde{\mathcal{E}} >= 0$ and $\nabla \times \tilde{\mathcal{E}} = 0$. It is also true that

$$< \frac{1}{\epsilon}(\epsilon^*)^{-1}\mathcal{D} \cdot (\epsilon^*)^{-1}\mathcal{D} >= (\epsilon^*)^{-2} < \frac{1}{\epsilon}\mathcal{D} \cdot \mathcal{D} > \tag{3.10}$$

$$= (\epsilon^*)^{-2} < \epsilon\mathcal{E} \cdot \mathcal{E} >$$

$$= (\epsilon^*)^{-2} < \mathcal{D} \cdot e >= (\epsilon^*)^{-1}$$

Hence the two variational problems are consistent.

The construction of bounds for ϵ^* reduces to selecting suitable trial fields $\mathcal{E}, \mathcal{D}$ to insert into the variational problems. Choosing $\mathcal{E} = e, \mathcal{D} = e$ leads to the Wiener bounds

$$(< \frac{1}{\epsilon} >)^{-1} \leq \epsilon^* \leq \quad < \epsilon >$$

known since early in the century (Bergmann 1978).

As pointed out by Kohler and Papanicolaou (1982), there are two basic considerations to the construction of bounds:

1. Utilize the underlying structure of the medium ("its geometry") in the analysis, e.g., dielectric spheres embedded in a homogeneous medium, to produce *geometry dependent bounds*; or

2. Produce bounds which are *geometry independent* for materials in which the components have a complex infrastructure.

The Claussius–Mossotti formulas are examples of bounds of the first kind. The results obtained using the Coherent Potential Approximation in the previous section are examples of geometry dependent bounds.

3.2.1 Geometry Independent Bounds

As in the previous section, we introduce a parameter ϵ_r which is the dielectric constant of a "reference medium" and rewrite Maxwell's equations as

$$\nabla \cdot (\epsilon_r E) + \nabla \cdot [(\epsilon - \epsilon_r) \cdot E] = 0, \quad \nabla \times E = 0 \tag{3.11}$$

Introducing the abstract operator

$$\Gamma E = -\nabla(-\Delta)^{-1}(\nabla \cdot E) \tag{3.12}$$

we can rewrite (3.11) as

$$E + \Gamma(\epsilon^{-1}(\epsilon - \epsilon_r)E - < \epsilon^{-1}(\epsilon - \epsilon_r)E >) = \bar{E} \tag{3.13}$$

(which actually encompasses all of Maxwell's equations). As pointed out in (Kohler and Papanicoloau 1982), the operator Γ is a projection on the family of zero mean, stationary random fields into the subspace of curl-free random fields. Formally,

$$\Gamma^2 = \Gamma$$

In addition, if $\mathcal{E}$ is a stationary random field which has zero mean and is *isotropic*, i.e., the correlation function satisfies

$$< \mathcal{E}_i(x + y)\mathcal{E}_j(y) >= R_{ij}(|x|)$$

then

$$< \Gamma\mathcal{E} \cdot \mathcal{E} >= \frac{1}{3} < \mathcal{E} \cdot \mathcal{E} > \tag{3.14}$$

To associate (3.13) with a variational problem, we follow (Kohler and Papanicolaou 1982) and introduce the polarization field

$$P = \epsilon_r^{-1}(\epsilon - \epsilon_r)E \tag{3.15}$$

which satisfies

$$\epsilon_r(\epsilon - \epsilon_r)^{-1}P + \Gamma(P - < P >) = \bar{E} \tag{3.16}$$

Equation (3.16) is the Euler equation for the quadratic functional

$$U(\mathcal{P}) = \frac{1}{2} < \epsilon_r(\epsilon - \epsilon_r)^{-1}\mathcal{P} \cdot \mathcal{P} > \tag{3.17}$$

$$+\frac{1}{2} < \Gamma(\mathcal{P} - < \mathcal{P} >) \cdot (\mathcal{P} - < \mathcal{P} >) > - < e \cdot \mathcal{P} >$$

When $\epsilon_r < \epsilon(x,\omega), \forall(x,\omega)$, then $U(\mathcal{P})$ achieves its minimum at P the solution of (3.16) and

$$U(P) = \frac{1}{2}\left(1 - \frac{\epsilon^*}{\epsilon_r}\right) \tag{3.18}$$

Thus,

$$\frac{1}{2}\left(1 - \frac{\epsilon^*}{\epsilon_r}\right) \leq U(\mathcal{P}), \text{ when } (\epsilon - \epsilon_r) > 0$$

for any trial field $\mathcal{P}$. This bound can be rewritten as

$$\epsilon_r(1 - 2U(\mathcal{P})) \leq \epsilon^*, \text{ when } (\epsilon - \epsilon_r) > 0 \tag{3.19}$$

When $(\epsilon - \epsilon_r) < 0$ the quadratic form $U(\mathcal{P})$ is negative definite. This provides the upper bound

$$\epsilon^* \leq \epsilon_r(1 - 2U(\mathcal{P})), \text{ when } (\epsilon - \epsilon_r) < 0 \tag{3.20}$$

The objective of the analysis is to choose trial fields $\mathcal{P}$ to make the bounds (3.19)(3.20) as tight as possible. Rewriting (3.15) as

$$P = \frac{\epsilon - \epsilon_r}{\epsilon_r}[e - \Gamma(P - < P >)] \tag{3.21}$$

there are several different ways to proceed.

First, if the medium is *nearly uniform:* $|\delta\epsilon| \triangleq |\epsilon - \epsilon_r|$ is small for some ϵ_r, then we can expand P in a power series in $\delta\epsilon$, and produce trial fields by truncating the series; e.g.,

$$\mathcal{P} = \frac{\delta\epsilon}{\epsilon_r}e - \frac{\delta\epsilon}{\epsilon_r}\Gamma\left(\frac{\delta\epsilon}{\epsilon_r}e - < \frac{\delta\epsilon}{\epsilon_r}e >\right) \tag{3.22}$$

This case has been treated extensively; see (Bergmann 1978) (Milton 1981a,b) (Golden and Papanicolaou 1983).

Second, if $\delta\epsilon$ is not small, but the medium has a *host-inclusion* geometry. Then the multiple scattering formalism can be used to derive bounds. This was illustrated in the previous section. We shall consider it further in the next subsection.

In the remainder of this subsection we shall consider the third alternative - $\delta\epsilon$ is not small for any ϵ_r and the medium is not of the simple host-inclusion type. However, the medium (the random function $\epsilon(x,\omega)$) is *statistically isotropic.*

Introducing

$$\Gamma_0 = \frac{1}{3}I \text{ and } \Gamma_1 = \Gamma - \Gamma_0$$

we have

$$< \Gamma_1\mathcal{E}_1 \cdot \mathcal{E}_2 >= 0, \quad < \Gamma\mathcal{E}_1 \cdot \mathcal{E}_2 >= \frac{1}{3} < \mathcal{E}_1 \cdot \mathcal{E}_2 >$$

for any fields $\mathcal{E}_i, i = 1,2$ that are isotropic. Using this notation, we can rewrite (3.15) in the form

$$P- < P > + \frac{\delta\epsilon}{\epsilon_r}\Gamma_0(P- < P >) + \frac{\delta\epsilon}{\epsilon_r}\Gamma_1(P- < P >) = \frac{\delta\epsilon}{\epsilon_r}e- < P >$$

or

$$\left[I + \left(I + \frac{\delta\epsilon}{\epsilon_r}\Gamma_0\right)^{-1}\frac{\delta\epsilon}{\epsilon_r}\Gamma_1\right](P- < P >) = \left(I + \frac{\delta\epsilon}{\epsilon_r}\Gamma_0\right)^{-1}\left[\frac{\delta\epsilon}{\epsilon_r}e- < P >\right] \tag{3.23}$$

Defining

$$L = -\left(I + \frac{\delta\epsilon}{\epsilon_r}\Gamma_0\right)^{-1}\frac{\delta\epsilon}{\epsilon_r}\Gamma_1$$

we can write (3.23) in the form

$$P- < P >= \sum_{n=1}^{\infty} L^n\left[\left(I + \frac{\delta\epsilon}{\epsilon_r}\Gamma_0\right)^{-1}\left(\frac{\delta\epsilon}{\epsilon_r}e- < P >\right)\right] \tag{3.24}$$

We can choose trial fields by taking successive terms in this series. The simplest is

$$\mathcal{P}- < \mathcal{P} >= \left(I + \frac{\delta\epsilon}{\epsilon_r}\Gamma_0\right)^{-1}\left(\frac{\delta\epsilon}{\epsilon_r}e- < \mathcal{P} >\right) \tag{3.25}$$

Taking averages in (3.25), and using the definition of Γ_0, we have

$$< \mathcal{P} >= \left[< \left(I + \frac{\delta\epsilon}{3\epsilon_r}\right)^{-1} >\right]^{-1} < \left(I + \frac{\delta\epsilon}{3\epsilon_r}\right)^{-1}\frac{\delta\epsilon}{\epsilon_r} > e \tag{3.26}$$

and

$$U(\mathcal{P}) = -\frac{1}{2} < \mathcal{P} > \cdot e \tag{3.27}$$

Using (3.26)(3.27) in (3.19)(3.20) we have

$$\epsilon_r\left\{1 + \left[< \left(1 + \frac{\delta\epsilon}{3\epsilon_r}\right)^{-1} >\right]^{-1} < \left(1 + \frac{\delta\epsilon}{3\epsilon_r}\right)^{-1}\frac{\delta\epsilon}{\epsilon_r} >\right\} \leq \epsilon^* \tag{3.28}$$

when $\delta\epsilon > 0$

and

$$\epsilon^{*} \leq \epsilon_r \left\{ 1 + \left[< \left(1 + \frac{\delta\epsilon}{3\epsilon_r} \right)^{-1} > \right]^{-1} < \left(1 + \frac{\delta\epsilon}{3\epsilon_r} \right)^{-1} \frac{\delta\epsilon}{\epsilon_r} > \right\} \tag{3.29}$$

when $\delta\epsilon < 0$

Note that these bounds apply to general media which have a statistically isotropic structure.

When the medium has only two components - $\epsilon(x,\omega)$ takes values ϵ_0, ϵ_0 with $0 < \epsilon_0 < \epsilon_1 < \infty$, then inequality (3.28) is optimized by taking $\epsilon_r = \epsilon_0$; and inequality (3.29) is optimized by taking $\epsilon_r = \epsilon_1$. This leads to the classical bounds of Hashin and Shtrikman (1962)

$$\epsilon_0 + \frac{\rho_1}{(1/(\epsilon_1 - \epsilon_0) + \rho_0/3\epsilon_0} \leq \epsilon^{*} \leq \epsilon_1 + \frac{\rho_0}{(1/(\epsilon_0 - \epsilon_1) + \rho_1/3\epsilon_1} \tag{3.30}$$

As before, $\rho_i, i = 0,1$ is the volume fraction occupied by material of type i.

By takng additional terms in the expansion (3.24) one can obtain better trial fields. Note, however, that each succeeding trial field requires even more statistical information. For example, the next trial field corresponding to keeping terms $n = 0,1$ in (3.24) requires second order statistics (e.g., correlation functions) on the random media.

3.2.2 Geometry Dependent Bounds

Now suppose that material is of the host-inclusion type with a specific geometry, spheres with dielectric constants $\epsilon_i, i = 1,\ldots,m$ embedded in a uniform background

material. Thus, let $y_{ij}, i = 1, \ldots, m, j = 1, \ldots, N_i$ be the (random) locations of centers of the spheres, each of radius δ. Assume that the spheres do not overlap. Then

$$\epsilon(x,\omega) = \begin{cases} \epsilon_i, & |x - y_{ij}| \leq \delta, \quad \text{for some } i,j \\ \epsilon_0, & |x - y_{ij}| > \delta, \quad \text{for all } i,j \end{cases} \tag{3.31}$$

As before, $\rho_i = \frac{4}{3}\pi\delta^3 c_i$ is the volume fraction occupied by the spheres or class i, where c_i is the average number of sphere centers per unit volume.

To construct suitable trial fields, we use the transition operator - multiple scattering formalism. Recall the formulation of Maxwell's equations for the multicomponent medium

$$\nabla \cdot (\epsilon_0 E) + \sum_{i=1}^{m} \sum_{j=1}^{N_i} \nabla \cdot \left[(\epsilon_i - \epsilon_0) \chi_{v_{ij}} E \right] = 0 \tag{3.32}$$

$$\nabla \times E = 0$$

which we write in abstract form as

$$(L_0 + M)E = 0, \quad \nabla \times E = 0 \tag{3.33}$$

As before

$$E + L_0^{-1} M E = F \tag{3.34}$$

where F satsifies $L_0 F = 0, < E >= \bar{E}$. In terms of the transition operator $T = (L_0 + M)^{-1} M$ we may write this as

$$\begin{aligned} E &= (I - T)F \\ &= (I - \sum_{i=1}^{m} T_i)F \end{aligned} \tag{3.35}$$

$$= (I - \sum_{i=1}^{m} \sum_{j=1}^{N_i} T_{ij})F$$

Based on this we define the trial field

$$\mathcal{E} = F - \sum_{i=1}^{m} \sum_{j=1}^{N_i} T_{ij}F \tag{3.36}$$

with F a constant to be chosen. Writing out the operator T_{ij} explicitly we have

$$\mathcal{E} = F - \sum_{i=1}^{m} \left(\frac{\epsilon_i - \epsilon_0}{2\epsilon_i + \epsilon_0} \right) \sum_{j=1}^{N_i} \left\{ \left(1 - \chi_{v_{ij}}(x) \right) \delta^3 \nabla \left[\frac{(x - y_{ij}) \cdot F}{|x - y_{ij}|^3} \right] + \chi_{v_{ij}}(x) F \right\} \tag{3.37}$$

This trial field is curl free. We choose F so that $< \mathcal{E} >= \bar{E}$. If we take $\{y_{ij}\}$ to be Poisson process with parameter c, and if we assume a suitable cutoff region so the series converges,[3] then

$$\bar{E} = F - \sum_{i=1}^{m} \left(\frac{\epsilon_i - \epsilon_0}{2\epsilon_i - \epsilon_0} \right) \frac{4}{3} \pi \delta^3 c F \tag{3.38}$$

Hence,

$$F = \left(1 - \sum_{i=1}^{m} \rho_i \frac{\epsilon_i - \epsilon_0}{2\epsilon_i - \epsilon_0} \right)^{-1} \bar{E} \tag{3.39}$$

Now let

$$\gamma_i = \frac{\epsilon_i - \epsilon_0}{2\epsilon_i - \epsilon_0}, \quad i = 1, \ldots, m$$

$$E_i^{(1)}(x) = \gamma_i \delta^3 \nabla \left(\frac{x \cdot \bar{E}}{|x|^3} \right), \quad E_i^{(2)} = \gamma_i \bar{E}$$

then (3.37)

$$\mathcal{E} = \left(1 - \sum_{i=1}^{m} \rho_i \gamma_i \right)^{-1} \left[\bar{E} - \sum_{i=1}^{m} \sum_{j=1}^{N_i} \{ E^{(1)}(x - y_{ij})(1 - \chi_\delta(x - y_{ij})) \right. \tag{3.40}$$

$$\left. + E_i^{(2)}(x - y_{ij}) \chi_\delta(x - y_{ij}) \} \right]$$

[3]This is the case of the ATA divergence.

where χ_δ is the characteristic function of the sphere of radius δ centered at the origin.

For the case $m = 1$ (one class of embedded scatterers), this formula yields the upper bound (Kohler Papanicolaou 1981) (using the Poisson distribution for the locations of the scattering centers)

$$\epsilon^* \leq \frac{1 + \kappa\rho - 2\gamma(\rho + \kappa\rho + \kappa\rho^2) + \gamma^2[5\rho + 7\kappa\rho^2 + \kappa\rho + \rho^2 + \kappa\rho^3]}{(1 - \gamma\rho)^2} \tag{3.41}$$

where $\gamma = \gamma_1, k = \epsilon_1/\epsilon_0, \kappa = k - 1$, and $\rho = \rho_1$.

A lower bound can be obtained by a similar procedure, starting from the complementary variational principle. The trial field is computed again by expanding the expression for T_{ij}. The resulting field is

$$\mathcal{D} = G - \sum_{i=1}^{m}\sum_{j=1}^{N_i}\left\{(1 - \chi_{v_{ij}}(x))\gamma_i\delta^3\nabla\left(\frac{(x - y_{ij})\cdot G}{|x - y_{ij}|^3}\right) - 2\gamma_i\chi_{v_{ij}}(x)G\right\} \tag{3.42}$$

which satisfies $< \mathcal{D} >= \bar{D}$, assuming the elementary scatterers do not overlap. To have $< \mathcal{D} >= \bar{D}$, we must have

$$G = (1 + 2\sum_{i=1}^{m}\gamma_i\rho_i)^{-1}\bar{D}$$

In the case $m = 1$, direct computation of the average $< \epsilon\mathcal{D}\cdot\mathcal{D} >$ yields the estimate

$$(\epsilon^*)^{-1} \leq \frac{1}{\epsilon_0}\frac{1 + \kappa_1\rho - 2\gamma(\rho + \kappa_1\rho - 2\kappa_1\rho^2) + \gamma^2(8\rho + 16\kappa_1\rho^2 + 4\rho^2 + 4\kappa_1\rho^3)}{(1 + 2\gamma\rho)^2} \tag{3.43}$$

where γ is as above, and $\kappa_1 = k^{-1} - 1, k = \epsilon_1/\epsilon_0$.

Improved bounds may be obtained using the Coherent Potential Approximation to compute the expression for T_{ij}. It is not necessary to repeat the computation. Rather this may be accomplished simply by introducing the reference dielectric constant ϵ_r.

The trial fields are constructed by assuming that the exterior dielectric constant is ϵ_r and that the dielectric constant inside the scatterers is $\epsilon_r + \epsilon_i - \epsilon_0$ rather than ϵ_i. This changes the trial fields, altering the parameter γ_i to

$$\gamma_i = \frac{\epsilon_i - \epsilon_0}{3\epsilon_r + \epsilon_i - \epsilon_0} \tag{3.44}$$

Now γ_i varies on the interval

$$\frac{k_i - 1}{4k_i - 1} < \gamma_i < \frac{k_i - 1}{k_i + 2} \tag{3.45}$$

where $k_i = \epsilon_i/\epsilon_0$ and we assume $0 \leq \epsilon_0 \leq \epsilon_i$.

3.3 Bounds Based on the Method of Analytic Continuation

In this subsection we shall examine the analytical continuation approach to deriving bounds, indicate its range of applicability, and discuss techniques for adapting the method to treat scattering and absorption from foliage. The method is based on writing the effective dielectric constant as a function of the (complex) dielectric constants of the underlying component media. Hence, it is represented as an analytic function of a complex variable, and the properties of such functions can be used to derive bounds on the effective parameter.

3.3.1 Bounds for Two and Three Component Media

If we substitute (3.1) into (3.5) and divide by ϵ_N, then we obtain

$$m_{ik}(h_1, \ldots, h_{N-1}) = \frac{\epsilon^*_{ik}}{\epsilon_N} = \int_\Omega P(d\omega) \left(\sum_{j=1}^{N-1} h_j \chi_j(\omega) + \chi_N \right) E_i^k(\omega) \tag{3.46}$$

Here $h_i = \epsilon_i/\epsilon_N$. The key idea in the work of Bergman (Bergmann 1978) is to regard $m(\cdot)$ as a function of the complex variables $h_1, \ldots, h_{N-1}$ in C^{N-1}. This permits one to use (spectral) representation formulas for $m(\cdot)$ which allow the derivation of bounds based on the extreme points of certain sets of measures. By expanding the representation formulas about the case of a "homogeneous" medium $h_j = 1, j = 1, \ldots, N-1$, it is possible to obtain a characterization of the underlying measures in terms of their "moments." These moments are functions of increasing amounts of information on the statistical properties of the medium. For example, if only the volume fractions ρ_j from (3.2) are known, then only the first term in the representation can be computed explicitly. This leads to the classical Wiener bounds:

For two-component media -

$$\left(\frac{\rho_1}{\epsilon_1} + \frac{\rho_2}{\epsilon_2}\right)^{-1} \leq \epsilon^* \leq \rho_1\epsilon_1 + \rho_2\epsilon_2 \tag{3.47}$$

For three-component media -

$$\left(\frac{\rho_1}{\epsilon_1} + \frac{\rho_2}{\epsilon_2} + \frac{\rho_3}{\epsilon_3}\right)^{-1} \leq \epsilon^* \leq \rho_1\epsilon_1 + \rho_2\epsilon_2 + \rho_3\epsilon_3 \tag{3.48}$$

These bounds are achieved by parallel plane configurations of the materials.

If additional information is available on the statistical description of the medium, then more sophisticated bounds can be obtained. For example, for a two-component medium which is statistically isotropic, the first moment of the spectral measure can be computed (in the expansion). This leads to the Hashin-Shtrikman bounds (assume without loss of generality $\epsilon_1 \leq \epsilon_2$)

$$\epsilon_1 + \frac{\rho_2}{1/(\epsilon_2 - \epsilon_1) + \rho_1/3\epsilon_1} \leq \epsilon^* \leq \epsilon_2 + \frac{\rho_1}{1/(\epsilon_1 - \epsilon_2) + \rho_2/3\epsilon_2} \tag{3.49}$$

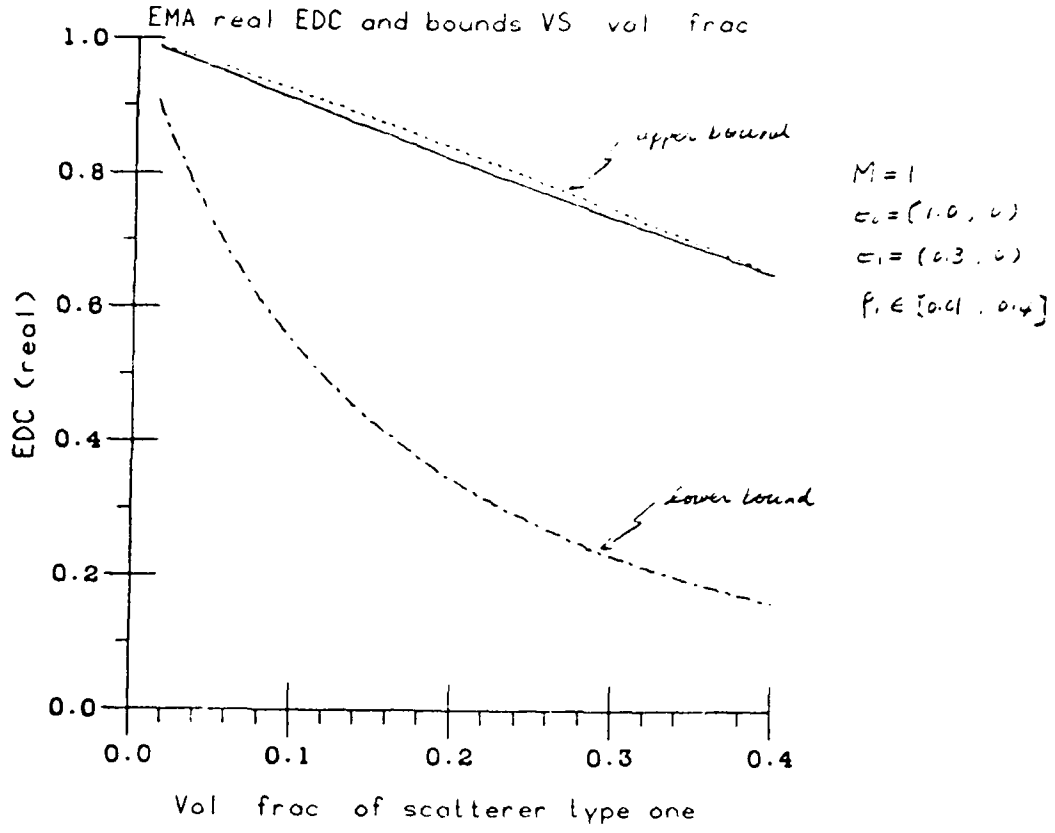

Figure 3.1: Effective permittivity from EMA versus bound estimates.

In Figure 3.3.1 we show a typical result illustrating the relationship between the upper and lower bounds and the effective permittivity (real) versus volume fraction as computed by the T-matrix EMA method.

3.3.2 Bounds for Multicomponent Media

There is a substantial difficulty in passing from two-component materials to N-component materials with $N \geq 3$ when using the analytical continuation method. Basically, one must deal with functions of more than one complex variable $(h_1, h_2, \ldots)$. The precise difficulty arises from the fact that the extreme points of the associated set of spectral measures are not known. Prior to the paper (Golden Papanicolaou 1985) there was no systematic method for treatment of multi-component media (using the continuation method). In the case of three-component media, it is possible to circumvent this problem by imposing a linear relationship among the complex parameters, and expanding the spectral measure about the (linear) parameter. In (Golden Papanicolaou 1985) a technical hypothesis was introduced which permitted recovery of the Hashin-Shtrikman bounds for three-component media. Assuming $\epsilon_1 \leq \epsilon_2 \leq \epsilon_3$ we have

$$\epsilon_1 + 1/\left(\frac{1}{A_1} - \frac{1}{3\epsilon_1}\right) \leq \epsilon^* \leq \epsilon_3 + 1/\left(\frac{1}{A_3} - \frac{1}{3\epsilon_3}\right) \tag{3.50}$$

$$A_j = \sum_{i=1} 3\rho_i / \left(\frac{1}{\epsilon_i - \epsilon_j} + \frac{1}{3\epsilon_j}\right)$$

These bounds are "optimal" for materials consisting a mixture of spheres of all different sizes of ϵ_1 and ϵ_2 materials each coated with ϵ_3 material in the appropriate volume fraction. Additional bounds were obtained in (Golden 1986) for complex permittivities (thus including the absorption of energy in the material).

The analytical continuation method is a powerful technique for derivation of bounds on the material properties of multi-component media; and it is useful to examine its potential for the evaluation of scattering/absorption phenomena in the interaction of EM radiation and foliage. The following points are important:

- The method is based on the assumption that the medium is nearly homogeneous; that is, the normalized permittivities $\epsilon_j/\epsilon_N \approx 1$. This is a restrictive assumption for foliage, and it may limit application of the approximation to certain kinds of foliage (mixtures of grasses, etc.).

- While the case of two component media is reasonably well developed; the case of three-component media is less clear. This is the key case if one is to understand the interaction of scattering effects in an environment of two different types of scatterers in a uniform background. The limitations on the method are both technical and physical.

  - The technical limitations are mainly due to the incomplete characterization of the set of spectral measures for functions of two complex variables.

  - The physical limitations are more fundamental. The method does not account for the geometry of multiple scattering processes. That is, while it is capable, in principle, of including higher order statistical functions in the expansion for the effective permittivity, the method develops the bounds in a fashion which is well removed from the physical properties of the medium. The extreme points of the set of measures which are used to determine the bounds are singular measures ("delta functions") which may not correspond well to the distributions of scatterers in typical foliage. It would be better to develop the bounds in a (restricted) set of measures which are representative of foliage.

4 Homogenization and Multiple Scattering

In this section we turn our attention to an alternative method for representation of multiple scattering effects in heterogeneous media. The method is a variation of the "homogenization" procedure which has been used widely in mathematical physics and engineering to develop effective media approximations. This method has promise for for several reasons:

1. It leads more naturally than the T-operator formalism to a compact representation for an effective parameter (complex dielectric constant) representation for the effective medium approximation.

2. The resulting representation for the effective dielectric constant includes the interaction of microscopic effects (multiple scattering) explicitly.

3. The underlying analysis applies (however, with significant differences in detail) to both periodic and random media. In the former case it is possible to solve the equations for the approximation; in the latter it is necessary to develop approximations which use physically measurable quantities (second order statistics and correlation functions).

4. One can prove convergence of the scaled model to the "homogenized" model in both the periodic and random case.

5. The representation provides a (formal) basis for the systematic construction of a sequential approximation to the effective dielectric constant, including "higher

order" expansions for the effective parameters in terms of the small parameter, at least in the periodic case.

We wish to undertake a systematic investigation of such approximations in the context of scattering from foliage covered terrain, and the multiscale (homogenization) method offers a more general setting for such a comparison than does the T-matrix formalism, at least in the frequency range to which it applies.

4.1 A General Model

To illustrate the ideas, consider the following general model: Let $\mathcal{O} \subset \Re^3$ be a region in which $\epsilon(x), \mu(x)$ and $\sigma(x)$ are, respectively, the dielectric tensor, magnetic permeability tensor, and conductivity tensor (3×3 matrices) of the material in region $\mathcal{O}$. In $\mathcal{O}$ the (vector-valued) electric and magnetic fields satisfy

$$\frac{\partial}{\partial t}\begin{bmatrix} \epsilon(x) & 0 \\ 0 & \mu(x) \end{bmatrix}\begin{bmatrix} E(x) \\ H(x) \end{bmatrix} = \begin{bmatrix} 0 & \nabla\times \\ -\nabla\times & 0 \end{bmatrix}\begin{bmatrix} E(t,x) \\ H(t,x) \end{bmatrix} \tag{4.1}$$

$$+\begin{bmatrix} \sigma(x) & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} E(t,x) \\ H(t,x) \end{bmatrix}$$

$$E(0,x) = E_0(x), H(0,x) = H_0(x) \tag{4.2}$$

$$\nabla \cdot [\epsilon E_0] = 0, \nabla \cdot [\mu H_0] = 0 \tag{4.3}$$

where $\nabla\times$ is the curl operator and $\nabla\cdot$ = div. Note that (4.3) implies

$$\nabla \cdot [\epsilon E(t,x)] = 0, \nabla \cdot [\mu H(t,x)] = 0, \forall t \geq 0$$

It is necessary to *assume* that $\epsilon(x), \mu(x)$ and $\sigma(x)$ are *symmetric*, and that $\epsilon(x), \mu(x)$ are *positive definite* matrices.

Using

$$\nabla \times E = \begin{bmatrix} 0 & -\frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} & 0 & \frac{\partial}{\partial x_1} \\ -\frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} \tag{4.4}$$

we see that (4.1) may be written in the abstract form

$$A^0(x)\frac{\partial v}{\partial t} = \sum_{p=1}^{3} A^p(x)\frac{\partial v}{\partial x_p} + B(x)v(t,x) \tag{4.5}$$

where

$$v(t,x) = \begin{bmatrix} E(t,x) \\ H(t,x) \end{bmatrix} \in \Re^6 \tag{4.6}$$

$$A^0(x) = \begin{bmatrix} \epsilon(x) & 0 \\ 0 & \mu(x) \end{bmatrix}$$

the $A^p(t,x)$ are skew-symmetric, 6×6 matrices with 0's and 1's as elements, and

$$B(x) = \begin{bmatrix} \sigma(x) & 0 \\ 0 & 0 \end{bmatrix} \in \Re^{6\times 6}$$

The symmetry assumption means that (4.5) is a *symmetric - hyperbolic system* of a type common in mathematical physics. The (asymptotic) analysis of systems of this type with *periodic* coefficients depending on a small (dimensionless) parameter, e.g.,

$$A^0_\nu(x/\nu), A^p_\nu(x/\nu), B_\nu(x/\nu)$$

(for $0 < \nu << 1$) was carried out in (Bensoussan, Lions, and Papanicolaou 1978) and in (Sanchez-Palencia 1980) using multi-scale methods (homogerization). The periodic case represents media with regularly imbedded components.

The "inhomogeneities" in the media which govern the scattering of EM radiation from foliage covered terrain are randomly distributed, and so, the homogenization (multi-scale) method must be extended to treat this case. A general theory for partial differential equations with random coefficients was developed in (Papanicolaou and Varadhan 1979). This work forms the basis for our treatment of the random scattering (and absorption) problem.

4.2 Scaling the Model

To set up the analysis, we must prescribe a scaling for the parameters of the random medium, in this case the permittivity, permeability, and conductivity. The general form is

$$\begin{aligned} \epsilon(x) &= \epsilon^{\nu}(x) = \epsilon(x, x/\nu) \\ \mu(x) &= \mu^{\nu}(x) = \mu(x, x/\nu) \\ \sigma(x) &= \sigma^{\nu}(x) = \sigma(x, x/\nu) \end{aligned} \tag{4.7}$$

In this representation, $\nu > 0$ is a small ($<< 1$) dimensionless parameter which characterizes the relative scale on which microscopic variations in the medium take place. In previous reports we have shown how this parameter may be identified in terms of the

mean free path between elementary scatterers in a foliage covered area.[4]

The presence of two scales in (4.7) suggests that we are permitting variations of the parameters across the structure of two different spatial types. The "fast scale" $y = x/\nu$ captures the effects of microscopic variations; that is, as we pass from one elementary scatterer to the next. Variations on the "slow scale" x capture the effects of macroscopic variations; that is, variations as we move from one type of vegetation to another, or from one region of a given type to another. Note that as y changes by one unit, the macroscopic scale x changes by $\nu << 1$ units; and as x changes by one unit, y changes by $1/\nu >> 1$ units.

Using this scaling, there are two basic cases which can be treated by versions of the method of multiple scales:

1. The functions $\epsilon(x,y), \mu(x,y), \sigma(x,y)$ are periodic in their second argument.

2. The functions $\epsilon(x,y), \mu(x,y), \sigma(x,y)$ are random functions of their second argument which are stationary and satisfy a mixing condition.

The first case was treated in (Bensoussan, Lions, and Papanicolaou 1978) and in (Sanchez-Palencia 1980), (see also the work of Tartar and Murat (Tartar 1986)) among many other related problems. The second case provides a basis for modeling certain types of foliage covered terrain.

[4]That is, ν may be defined in terms of the mean free path between scattering centers normalized by a characteristic length of the interaction process, e.g., the length of a scattering region or the principal axis of the first Fresnel zone, etc.

The mixing which must be imposed in the second case is that the statistical correlations between random events in the medium must decrease rapidly over the volume of interest. That is, the correlation between values of $\epsilon(x, y_1)$ and $\epsilon(x, y_2)$ decreases as $|y_1 - y_2| \to \infty$. In effect, this means that the microstructure of the foliage at one point y_1 cannot depend too strongly on that at the (distant) point y_2. This is clearly reasonable in many situations. It may not, however, be the case in wind blown grass.

Notice that the scattering medium is still permitted to have (statistical or deterministic) variations which take place in the "macroscopic" x spatial scale.

4.3 The Multiple Scale Hypothesis

The *objective* of the multiple scale asymptotic analysis is to compute approximations to the fields $E^\nu(t, x), H^\nu(t, x)$ in the limit as $\nu \to 0$, that is, in the limit as the microscopic variations become increasingly dense.

A *formal procedure* for computing approximations is to assume an expansion of the fields in the form

$$E^\nu(x, x/\nu) = E_0(x) + \nu E_1(x, y) + \nu^2 E_2(x, y) + \cdots \tag{4.8}$$

with $y = x/\nu$ and a similar expansion for H^ν. Introducing the change of coordinates

$$\frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \frac{1}{\nu}\frac{\partial}{\partial y} \tag{4.9}$$

in Maxwell's equations, substituting the expansions for E^ν and H^ν, one finds a sequence of equations for the terms $E_k(x, y), k = 0, 1, 2, \ldots$ (and H_k). These may be solved using

a device called a "corrector" which is derived from a kind of "separation of variables" argument.

If the dissipation is zero ($\sigma = 0$), the result is a representation of the solution for the approximate field in the form

$$\frac{\partial}{\partial t}\begin{bmatrix} q_\epsilon(x) & 0 \\ 0 & q_\mu(x) \end{bmatrix}\begin{bmatrix} E_0(x) \\ H_0(x) \end{bmatrix} = \begin{bmatrix} 0 & \nabla\times \\ -\nabla\times & 0 \end{bmatrix}\begin{bmatrix} E_0(t,x) \\ H_0(t,x) \end{bmatrix} \tag{4.10}$$

$$E_0(0,x) = E_0(x), H_0(0,x) = H_0(x)$$

$$\nabla\cdot[q_\epsilon E_0] = 0, \nabla\cdot[q_\mu H_0] = 0$$

The (matrix-valued) functions $q_\epsilon(x), q_\mu(x)$, which depend on the macroscopic spatial scale, are defined in terms of $\epsilon(x,y), \mu(x,y)$ by the "corrected averages"

$$q_\alpha^{ij}(x) = \mathcal{M}\left[\alpha_{ij}\left(\delta_{ij} + \frac{\partial\chi_j^\alpha(y)}{\partial y_i}\right)\right], i,j = 1,2,3 \tag{4.11}$$

where α_{ij} is either ϵ_{ij} or μ_{ij}, and the function $\chi^\alpha(y)$ is the *corrector* associated with ϵ or μ. Note that the *effective parameters* in the approximation q_ϵ, q_μ are not just the averages of the rapid variations of the parameters $\epsilon(x,\cdot), \mu(x,\cdot)$ over the medium. They include the "correctors" which retain the microscopic interaction effects (multiple scattering) in the final approximation.

The correctors satisfy a system of the general form (see the following sections for details)

$$\nabla\cdot\left[\alpha(e_j + \nabla\chi_j^\alpha(y))\right] = 0, \quad \mathcal{M}[\chi_i] = 0 \tag{4.12}$$

where e_j is the j^{th} natural basis vector in $\Re^3$ In these equations $\mathcal{M}$ is the operation of averaging over a typical cell in the domain for the periodic case; and it is expectation

with respect to the stationary distribution describing the medium in the random case. When the conductivity $\sigma \neq 0$, the homogenized model is more complex, including frequency dependent effects – see below.

4.4 Implementation of the Approximation

Thus, to implement the approximation, we must solve the equations (4.12) for the correctors and then compute the averages in (4.11). Computation of the correctors is possible in the periodic case (see, e.g., (Begis, Duvaut, and Hassim 1981) (Bougat and Derieux 1978)). However, in the random case of interest here, the computation is very difficult. The equations (4.12) in the random case are defined path by path; hence, one would have to have a complete statistical characterization of the medium to be able to solve them. This is impossible in practice, and one must evaluate the correctors by a procedure like Monte Carlo simulations. Hence, these equations should be regarded as the basis for derivation of further approximations or alternative representations for the medium which require only physically attainable statistical information, e.g., second order statistics in the microstructure of the medium. Such approximations have been derived in other problems (conductivity, porosity, etc.) in the past. We shall consider these methods and their adaptation to the EM scattering/absorption problem in the next section.

Before giving the details of the derivation of the approximation, we shall make a few remarks on its interpretation.

- First, note that the general form of the approximation is valid (with a very different mathematical interpretation) in both the random and periodic cases.
- Second, the role of the correctors in providing the "correct" form of the approximation cannot be dismissed. Approximations which simply average the variations of the permittivity and permeability over a domain are incorrect - making an error of "order one" in the small parameter ν. That is, the approximation (4.11)(4.12) is an approximation of "order ν" (at least in the periodic case); omitting the correctors invalidates this estimate.

4.5 Derivation of the Effective Parameters Using Homogenization

There are two basic cases to consider:

1. The periodic case when the inhomogenieties in the medium are spatially periodic; and
2. The case when the inhomogenieties in the medium are randomly distributed.

The expressions for the effective parameters are easier to grasp in the periodic case; and they are more readily computable. For this reason we shall treat this case first as background to the random case.

4.5.1 Periodic Media

This problem was treated in (Bensoussan Lions and Papanicolaou 1978), (Sanchez-Palencia 1980), and (Tartar 1979). Our treatment is based on (Sanchez-Palencia 1980).

To begin, we rewrite Maxwell's equations

$$\frac{\partial D^\nu}{\partial t} = \nabla \times H^\nu - J^\nu + F \tag{4.13}$$

$$\frac{\partial B^\nu}{\partial t} = -\nabla \times E^\nu + G$$

$$E^\nu(x,0) = 0, \qquad H^\nu(x,0) = 0$$

where F, G are (localized) source terms and[5]

$$D_i^\nu = \epsilon_{ij}(\frac{x}{\nu})E_j^\nu, \quad B_i^\nu = \mu_{ij}(\frac{x}{\nu})H_j^\nu, \quad J_i^\nu = \sigma_{ij}(\frac{x}{\nu})E_j^\nu. \tag{4.14}$$

We assume that ϵ, μ, σ are smooth Y periodic functions of x/ν which are symmetric and positive definite.[6] Adopting the multiscale hypothesis

$$E^\nu(x,t) = E^0(x,t) + \nu E^1(x,y,t) + \cdots \tag{4.15}$$

$$H^\nu(x,t) = H^0(x,t) + \nu H^1(x,y,t) + \cdots$$

and similarly for D^ν, B^ν and J^ν with each term in the asymptotic expansion periodic in y. We shall use the change of coordinates

$$\frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \frac{1}{\nu}\frac{\partial}{\partial y} \tag{4.16}$$

[5] We use the *summation convention* throughout this section - repeated indices in an expression are summed.

[6] That is, $\epsilon_{ij} = \epsilon_{ji}, \quad \epsilon_{ij}\xi_i\xi_j \geq \gamma|\xi|^2, \gamma > 0.$

with $y = x/\nu$, and similarly

$$(\nabla \times \cdot) \to (\nabla_x \times \cdot) + \frac{1}{\nu}(\nabla_y \times \cdot)$$

$$\mathrm{div} \to \mathrm{div}_x + \frac{1}{\nu}\mathrm{div}_y.$$

Rewriting Maxwell's equations in terms of these coordinates, inserting the expansions (4.15) into them, and equating coefficients of like powers of ν, we arrive at a sequence of equations for $E^0, H^0, E^1, H^1, \ldots$. The $O(\nu^{-1})$ and $O(\nu^0)$ terms are

$$\nabla_y \times H^0 = 0, \nabla_y \times E^0 = 0 \tag{4.17}$$

$$\frac{\partial D^0}{\partial t} = \nabla_x \times H^0 + \nabla_y \times H^1 - J^0 + F \tag{4.18}$$

$$\frac{\partial B^0}{\partial t} = -\nabla_x \times E^0 - \nabla_y \times E^1 + G$$

The equations (4.17) imply that $H_0(x, y, t), E_0(x, y, t)$ are gradients in y for fixed values of the other arguments.

The terms $\nabla_y \times E^1$ and $\nabla_y \times H^1$ are derivatives of periodic (in y) functions; hence, they have zero mean with respect to the averaging operation

$$\mathcal{M}(\nabla_y \times E^1) \triangleq \frac{1}{|Y|}\int_Y \nabla_y \times E^1 dy = 0 \tag{4.19}$$

(and similarly for $\nabla_y \times H^1$) where Y is a typical "cell" of the periodic structure and $|Y|$ is the volume of the cell. Therefore, averaging the system (4.18) over the cell Y, we arrive at the *homogenized system*

$$\frac{\partial \bar{D}^0}{\partial t} = \nabla_x \times \bar{H}^0 - \bar{J}^0 + F \tag{4.20}$$

$$\frac{\partial \bar{B}^0}{\partial t} = -\nabla_x \times \bar{E}^0 + G$$

where the overbar denotes the averaging operation. To complete the model, we must derive the homogenized constituent equations corresponding to (4.14). These laws take two different forms, depending on the presence or absence of dissipation ($\sigma = 0$ or not).

Consider the divergence of (4.13)

$$\text{div} \left(\frac{\partial D^\nu}{\partial t} + J^\nu \right) = \text{div } F \tag{4.21}$$

$$\text{div} \left(\frac{\partial B^\nu}{\partial t} \right) = \text{div } G$$

Using the *anstaz* (4.15) and the coordinates (4.16), we have, at order ν^{-1},

$$\text{div}_y \left(\frac{\partial D^0}{\partial t} + J^0 \right) = 0 \tag{4.22}$$

and

$$\text{div}_y \left(\frac{\partial B^0}{\partial t} \right) = 0 \tag{4.23}$$

which, with the zero initial conditions, implies

$$\text{div}_y B^0 = \text{div}_y [\mu(x,y) H^0(x,y)] = 0 \tag{4.24}$$

From (4.17) we have that E^0 and H^0 are gradients; hence, we can write

$$E^0 - \bar{E}^0 = \nabla \phi, \qquad H^0 - \bar{H}^0 = \nabla \psi \tag{4.25}$$

with the means of ϕ and ψ zero. Using these in (4.22)(4.23), we have

$$\frac{\partial}{\partial y_i} \left\{ \mu_{ij}(y) \left[\frac{\partial \psi}{\partial y_j} + \bar{H}^0_j \right] \right\} = 0 \tag{4.26}$$

$$\frac{\partial}{\partial y_i} \left\{ \left[\epsilon_{ij}(y) \frac{\partial}{\partial t} + \sigma_{ij}(y) \right] \left[\frac{\partial \phi}{\partial y_j} + \bar{E}^0_j \right] \right\} = 0 \tag{4.27}$$

These are the key equations in application of the homogenization method to Maxwell's equations. If we regard (4.26) as an equation for the potential ψ, then, noting that $\bar{H}^0$ does not depend on y, we can solve (4.26) by assuming the "separation of variables"

$$\psi(x,y,t) = \chi_k^\mu(x,y,t)\bar{H}_k^0(x,t) \tag{4.28}$$

and regarding (4.26) as an equation for $\chi(y)$ as a function of y, treating x,t as fixed parameters. We call $\chi_k^\mu(x,y,t), k = 1,2,\ldots,d$ (d is the dimension of the space) the *correctors* associated with μ_{ij}. Using (4.26) and (4.28), the equation for the k^{th} corrector is

$$-\frac{\partial}{\partial y_i}\left[\mu_{ij}\frac{\partial \chi_k^\mu}{\partial y_j}\right] = \frac{\partial \mu_{ik}}{\partial y_i}, \quad k = 1,2,\ldots,d. \tag{4.29}$$

This problem plays a key role in the homogenization method. We call it the *cell problem*. The assumption that $\mu_{ij}(x,y)$ is periodic in y, symmetric, and positive definite (as a matrix for each x,y), guarantees that (4.29) has a unique solution[7] which has zero mean

$$\bar{\chi} = \frac{1}{|Y|}\int_Y \chi dy = 0$$

The solvability condition for (4.29) is that the right hand side have zero mean. This holds since μ_{ij} is periodic and smooth.

The homogenized form of the field equations may be obtained using the correctors. Writing out the $O(\nu^0)$ term in the expansion of (4.21), we have

$$\mathrm{div}_y B^1 + \mathrm{div}_x B^0 = \mathrm{div}_x G \tag{4.30}$$

[7]Among the set of functions periodic in y with first derivatives square integrable over a cell, i.e., periodic functions in $H^1_{loc}(\Re^d)$. See (Sanchez-Palencia 1980), pp. 51-54, or (Bensoussan Lions Papanicolaou 1978).

Using $\mathcal{M}$ to denote the averaging operation, the solvability condition for this equation is

$$\mathcal{M}\{\mathrm{div}_x B^0 - \mathrm{div}_x G\} = 0$$

which we rewrite as

$$calM\left[\mathrm{div}_x\{\mu(x,y)[\nabla_y\psi + \bar{H}^0(x)] - G\}\right] = 0 \tag{4.31}$$

Using the representation (4.28) for the potential, (4.31) holds, if

$$\mathcal{M}\left[\mu\left(\nabla_y\chi^\nu + \underline{1}\right)\bar{H}_0 - G\right] = 0.$$

This provides the definition of the *homogenized* magnetic permeability, μ^h, as

$$\mu^h_{ik} = \mathcal{M}\left(\mu_{ij}(x,y)\left[\delta_{ik} + \frac{\partial\chi^\mu_k}{\partial y_j}\right]\right) \tag{4.32}$$

The effective permittivity and conductivity will depend on the frequency of the incident radiation; and the analysis must reflect this. Recall (4.22), rewritten as

$$\mathrm{div}\left\{\frac{\partial}{\partial t}\epsilon E_0 + \sigma E_0\right\} = 0 \tag{4.33}$$

$$= \mathrm{div}\left\{\left[\frac{\partial}{\partial t}\epsilon + \sigma\right]\left[\nabla_y\phi + \bar{E}_0\right]\right\}$$

$$= \frac{\partial}{\partial t}\mathrm{div}_y\left[\epsilon\nabla_y\phi\right] + \mathrm{div}_y\left[\sigma\nabla_y\phi\right] \tag{4.34}$$

$$+\mathrm{div}_y\left[\epsilon\frac{\partial}{\partial t}\bar{E}_0\right] + \mathrm{div}_y\left[\sigma\bar{E}_0\right]$$

Thus, the local "cell" equation is an evolution (in t) equation.

To treat (4.34) we use a technique introduced in (Sanchez-Palencia and Sanchez-Hubert 1978). Let V_Y be the space of suitably smooth functions $\theta(y)$ which are periodic with zero mean, and let V_Y be endowed with the inner product

$$< \phi, \theta >_{V_Y} = \int_Y \epsilon_{ij} \frac{\partial \phi}{\partial y_i} \frac{\partial \theta}{\partial y_j} dy$$

Using this, (4.34) becomes

$$\frac{\partial}{\partial t} \int_Y \left[\epsilon_{ij} \frac{\partial \phi}{\partial y_j} + \epsilon_{ij} (\bar{E}_0)_j \right] \frac{\partial \theta}{\partial y_i} dy \tag{4.35}$$

$$+ \int_Y \sigma_{ij} \frac{\partial \phi}{\partial y_j} \frac{\partial \theta}{\partial y_i} dy + \int_Y \sigma_{ij} (\bar{E}_0)_j \frac{\partial \theta}{\partial y_i} = 0$$

Using the inner product notation, this may be written compactly as

$$\frac{\partial}{\partial t} < \phi + c_j^1 (\bar{E}_0)_j, \theta >_{V_Y} + < A\phi, \theta >_{V_Y} + < c_j^2 (\bar{E}_0)_j, \theta >_{V_Y} = 0 \tag{4.36}$$

where

$$A \approx \sigma_{ij} \frac{\partial}{\partial y_j}$$

$$< c_j^1, \theta >_{V_Y} = \int_Y \epsilon_{ij} \frac{\partial \theta}{\partial y_i} dy$$

$$< c_j^2, \theta >_{V_Y} = \int_Y \sigma_{ij} \frac{\partial \theta}{\partial y_i} dy$$

The evolution equation (4.36) can be solved for the potential ϕ

$$\phi(t) = -c_j^1 (\bar{E}_0)_j(t) + \int_0^t e^{-A(t-\tau)} c_j^3 (\bar{E}_0)_j(\tau) d\tau \tag{4.37}$$

$$c_j^3 = A c_j^1 - c_j^2$$

Using this expression for ϕ, we can compute the average flux vector $\bar{D}_0$ and average induced current $\bar{J}_0$ and define the effective permittivity and conductivity. Using $E_0 =$

$\bar{E}_0 + \nabla\phi$, we write the average of $D_0 = \epsilon E_0$ as

$$(\bar{D}_0)_i = \mathcal{M}\left\{\epsilon_{ij}(\bar{E}_0 + \nabla\phi)\right\} \tag{4.38}$$

$$= \mathcal{M}\left\{\epsilon_{ik}(\bar{E}_0)_k\right\}$$

$$+\mathcal{M}\left\{\epsilon_{ij}\left[\frac{\partial}{\partial y_k}\left(-c_j^1(\bar{E}_0)_j(t) + \int_0^t e^{-A(t-\tau)}c_j^3(\bar{E}_0)_j(\tau)d\tau\right)\right]\right\}$$

This may be written compactly as

$$(\bar{D}_0)_i = a_i^\epsilon j(\bar{E}_0)_j(t) + \int_0^t b_{ij}^\epsilon(t-\tau)(\bar{E}_0)_j(t-\tau)d\tau \tag{4.39}$$

where

$$a_{ij}^\epsilon = \mathcal{M}\left[\epsilon_{ij}(y) - \epsilon_{ik}(y)\frac{\partial c_j^1(y)}{\partial y_k}\right] \tag{4.40}$$

$$b_{ij}^\epsilon = \mathcal{M}\left[\epsilon_{ij}(y)\frac{\partial}{\partial y_i}\left(e^{-A\tau}c_j^3\right)(y)\right] \tag{4.41}$$

In a similar fashion, we can compute the induced current

$$(\bar{J}_0)_i = a_{ij}^\sigma(\bar{E}_0)_j(t) + \int_0^t b_{ij}^\sigma(t-\tau)(\bar{E}_0)_j(t-\tau)d\tau \tag{4.42}$$

where a^σ and b^σ are defined by (4.41)(4.42) with ϵ replaced by σ.

Using Laplace transforms, we can more readily express the effective scattering parameters as a function of frequency. Indeed, if we let $\hat{D}^\nu(\omega)$ be the Laplace transform of D^ν with respect to t, and similarly for the other variables, then Maxwell's equations take the form

$$\omega\hat{D}^\nu + \hat{J}^\nu = \nabla \times \hat{H}^\nu + \hat{F} \tag{4.43}$$

$$\omega\hat{B}^\nu = -\nabla \times \hat{E}^\nu + \hat{G}$$

The claim is that Maxwell's equations approach the homogenized laws

$$\omega \hat{D}^h + \hat{J}^h = (\omega\epsilon + \sigma)^h \hat{E}^h \tag{4.44}$$

$$\hat{B}^h = \mu^h \hat{H}^h$$

where the superscript $(\cdot)^h$ denotes the "homogenized" laws. Using the evolution equations (4.39)(4.42) we can identify

$$\omega a^{\epsilon}_{ij} + \omega \hat{b}^{\epsilon}_{ij}(\omega) + a^{\sigma}_{ij} + \hat{b}^{\sigma}_{ij}(\omega) = (\omega\epsilon_{ij} + \sigma_{ij})^h, \quad \Re\omega > 0 \tag{4.45}$$

This completes the derivation of the (first order) homogenization theory for Maxwell's equations.

Remark: Using the arguments in (Bensoussan Lions and Papnicolaou 1978) (Sanchez–Palencia and Sanchez–Hubert 1978) and (Sanchez–Palencia 1980), it can be shown that the homogenized Maxwell's equations (4.39)(4.42) (or the frequency domain version) have a unique solution. Moreover, it can be shown that the homogenized Maxwell's equations are the limits as $\nu \to 0$ of the original scaled Maxwell's equations (in an appropriate weak* topology).

Rather than give these arguments, we shall turn our attention to the case when the coefficients in Maxwell's equations are random processes in the spatial variables.

4.5.2 Random Media

As we have noted earlier, the formulas for the effective parameters in the homogenized form of Maxwell's equations carry over in form, at least, to the random case. The

averaging operation $\mathcal{M}$ is interpreted as expectation with respect to an appropriate probability measure. While the procedure of simply expanding the field quantities in asymptotic series in the small parameter is only formal in the random case, it can still be used with appropriate cautions to determine equations for the homogenized field quantities. The arguments for convergence of the scaled field quantities to the homogenized quantities are, however, substantially different in the random case.

The central difficulty in treating random coefficients arises in the meaning and evaluation of the "correctors" $\chi^\epsilon, \chi^\mu, \chi^\sigma$. They are random processes which must be evaluated (numerically) by a Monte Carlo type procedure. Before numerical methods are designed it is important to address the fundamental issue of existence and uniqueness of solutions to the corrector equations. Since each of the three correctors $\chi^\epsilon, \chi^\mu, \chi^\sigma$ satisfies the same type of elliptic equation with random coefficients, we shall treat the generic problem: Find $\chi(y;\omega)$ such that

$$\chi(0;\omega) = 0, \quad E\chi(y) = 0, \quad \forall y \tag{4.46}$$

$\partial\chi/\partial y_j$ is a square integrable stationary process $\forall j$ and

$$-\frac{\partial}{\partial y_i}\left(a_{ij}(y,\omega)\frac{\partial\chi}{\partial y_j}\right) = \frac{\partial g_j}{\partial y_j}(y,\omega)$$

in the sense of distributions. We assume that

$$a_0|\xi|^2 \le \sum_{i,j} a_{ij}(y,\omega)\xi_i\xi_j \le \frac{1}{a_0}|\xi|^2 \quad \forall \xi \in \Re^n, a_0 > 0 \tag{4.47}$$

where $a_{ij}(y,\omega)$ and $g_j(y,\omega)$ are square integrable stationary processes, $i,j = 1,\ldots n$.

We shall consider a set up for stationary processes as presented in (Papanicolaou and Varadhan 1979). Let $(\Omega, \mathcal{A}, P)$ be a probability space characterizing the uncertainties in the underlying medium, and define $\mathcal{H} = L^2(\Omega, \mathcal{A}, P)$. We assume that

$$\text{there exists a strongly continuous unitary group } T_y \text{ on } \mathcal{H}, y \in \Re^n \tag{4.48}$$

$$T_y \text{ is ergodic, which means if } \tilde{f} \in \mathcal{H} \text{ satisfies} \tag{4.49}$$

$$T_y \tilde{f} = \tilde{f}, \quad \forall y, \text{ then } \tilde{f} \text{ is a constant.}$$

$$\text{if } \tilde{f} \geq 0, \text{ then } T_y \tilde{f} \geq 0 \text{ and } T_y 1 = 1 \tag{4.50}$$

The group T_y, which is used to define a stationary processs on the probability space, has a spectral resolution defined by

$$T_y = \int_{\Re^n} e^{i\lambda y} U(d\lambda)$$

where $U(d\lambda)$ is a projection valued measure. We next define

$$D_i \tilde{f}(\omega) = \frac{\partial}{\partial y_i} \left(T_y \tilde{f}\right)(\omega)|_{y=0} \tag{4.51}$$

which are closed densely defined linear operators with domains $\mathcal{D}(D_i)$ in $\mathcal{H}$. Let $\mathcal{H}^1 = \bigcap_{j=1}^n \mathcal{D}(D_j)$ which is dense in $\mathcal{H}$. We equip $\mathcal{H}^1$ with the Hilbert scalar product

$$((\tilde{f}, \tilde{g}))_{\mathcal{H}^1} = E\tilde{f}\tilde{g} + \sum_{j=1}^{d} E D_j \tilde{f} D_j \tilde{g} \tag{4.52}$$

We identify $\mathcal{H}$ with its dual and call $\mathcal{H}^{-1}$ the dual of $\mathcal{H}^1$. We have the inclusions

$$\mathcal{H}^1 \subset \mathcal{H} \subset \mathcal{H}^{-1}$$

each space being dense in the next one with continuous injection. The family T_y is also a strongly continuous unitary group on $\mathcal{H}^1$.

Remark *The periodic case.*

Let Ω be the unit n dimensional torus, $\mathcal{A}$ the σ-algebra of Lebesgue measurable sets and P Lebesgue measure on Ω. Then $\mathcal{H}$ is the space of measurable periodic functions (period 1 in each component) such that

$$\int_\Omega (\tilde{f}(\omega))^2 d\omega < \infty.$$

We define

$$T_y \tilde{f}(\omega) = \tilde{f}(\omega + y)$$

hence

$$D_i \tilde{f} = \frac{\partial}{\partial \omega_i} \tilde{f}.$$

An important fact in the periodic case which does not carry over to the random case, is that there is no analogue of Poincare's inequality (Bensoussan, Lions, and Papanicolaou 1979).

Consider now random variables not necessarily in $\mathcal{H}$. We assume that

T_y is a linear group on the set of complex random variables such that $\forall \tilde{\eta}_1 \ldots \tilde{\eta}_k$ complex random variables, ψ Borel bounded function on C^k, then

$$E\phi(T_y\tilde{\eta}_1, \ldots, T_y\tilde{\eta}_k) = E\phi(\tilde{\eta}_1, \ldots, \tilde{\eta}_k) \tag{4.53}$$

$y, \omega \to T_y\tilde{\eta}$ is measurable,

$$T_y\tilde{\eta} \geq 0 \text{ if } \tilde{\eta} \geq 0.$$

A *stationary process* is a stochastic process $\eta(y;\omega)$ which can be represented in the form

$$\eta(y;\omega) = T_y\tilde{\eta}(\omega). \tag{4.54}$$

The space of square integrable stationary processes can be identified with $\mathcal{H}$. Moreover,

$$\frac{\partial \eta}{\partial y_i}(y;\omega) = D_i T_y \tilde{\eta}(\omega) = T_y D_i \tilde{\eta} \tag{4.55}$$

$$\text{if } \tilde{\eta} \in \mathcal{H}^1.$$

Note that the continuity assumption on T_y implies that the square integrable stationary processes are necessarily continuous function of y with values in $\mathcal{H}$. Hence, if $\tilde{\eta} \in \mathcal{H}$, $\eta(y;\omega) \in C^0(\Re^n;\mathcal{H})$, space of uniformly continuous functions on $\Re^n$ with values in $\mathcal{H}$. If $\tilde{\eta} \in \mathcal{H}^1$ then $\eta(y;\omega) \in C^1(\Re^n;\mathcal{H})$.

The Cell Problem We consider here stationary processes $a_{ij}(y;\omega)$ such that

$$a_0|\xi|^2 \leq \sum_{i,j} a_{ij}(y,\omega)\xi_i\xi_j \leq \frac{1}{a_0}|\xi|^2 \quad \forall \xi \in \Re^n, a_0 > 0 \tag{4.56}$$

Let $g_j(y,\omega) = T_y\tilde{g}_j$ square integrable stationary processes, $j = 1,\ldots n$. We shall solve the problem: Find

$$\chi(y;\omega) \in C^1(\Re^n;\mathcal{H}), \chi(0;\omega) = 0, E\chi(y) = 0 \quad \forall y \tag{4.57}$$

$$\partial\chi/\partial y_j \text{ is a square integrable stationary process} \quad \forall j$$

$$-\frac{\partial}{\partial y_i}\left(a_{ij}(y,\omega)\frac{\partial \chi}{\partial y_j}\right) = \frac{\partial g_j}{\partial y_j}(y,\omega)$$

in the sense of distributions with values in $\mathcal{H}$ (or as continuous functions with values in $\mathcal{H}^{-1}$).

Papanicolaou and Varadhan (1979) have shown the existence and uniqueness of the solution of (4.57). We shall follow their proof with minor changes. Note that $\chi(y;\omega)$ itself is *not* a stationary process. This is a big difference with respect to the periodic case and relates to the **Remark**. Note also that $\chi(y;\omega) \in C^2(\Re^n;\mathcal{H}^{-1})$.[8]

Let $\tilde{\chi}_j \in H$ such that

$$\frac{\partial \chi}{\partial y_j}(y;\omega) = T_y \tilde{\chi}_j$$

we can assert that

$$E\tilde{\chi}_j D_k \tilde{\phi} = E\tilde{\chi}_k D_j \tilde{\phi}, \quad \forall \tilde{\phi} \in \mathcal{H}^1. \tag{4.58}$$

Indeed we have to show that

$$D_k \tilde{\chi}_j = D_j \tilde{\chi}_k \tag{4.59}$$

as an equality in $\mathcal{H}^{-1}$. But

$$T_y D_k \tilde{\chi}_j = \frac{\partial}{\partial y_k} \chi_j(y;\omega)$$

$$= \frac{\partial^2 \chi}{\partial y_j \partial y_k}(y;\omega)$$

hence

$$T_y D_k \tilde{\chi}_j = T_y D_j \tilde{\chi}_k$$

which implies (4.59).

We have also

[8] By virtue of (4.50) it is sufficient to have (4.56) for $y = 0$.

$$E\dot{\chi} = 0. \tag{4.60}$$

This follows from

$$\chi(y;\omega) = \sum_j \int_0^1 T_{\theta y}\tilde{\chi}_j y_j d\theta$$

hence

$$E(\chi/y) = \sum_j E\tilde{\chi}_j y_j = 0$$

by the assumption. Therefore (4.60) follows.

The main result in this section is the following:

Theorem. *There exists one and only one solution of the corrector equation* (4.57).

This means that the problem of computing the effective parameter for the homogenization problem by averaging the coefficient $a_{ij}(x,\omega)$ with respect to the underlying probability measure is well–posed. See (Bensoussan and Blankenship 1988) for an application of this result in homogenization of nonlinear, stochastic control problems.

Proof.

Uniqueness.

Assume that $\partial g_j/\partial y_j = 0$. Define

$$\tilde{\phi}^\beta(\omega) = \int_{\mathbb{R}^n} \sum_j \frac{(-i\lambda_j - \beta)}{|i\lambda - \beta|^2} U(d\lambda)\tilde{\chi}_j(\omega). \tag{4.61}$$

Note that (4.57) can be written as

$$-D_i T_y(\tilde{a}_{iy}\tilde{\chi}_j) = 0 \text{ in } \mathcal{H}^{-1}$$

hence, as is easily seen

$$E\tilde{a}_{ij}\tilde{\chi}_j D_i\tilde{\phi}^\beta = 0. \tag{4.62}$$

Because of (4.59) we have

$$\int_{\mathbb{R}^n} \frac{\lambda_k U(d\lambda)\tilde{\chi}_j}{|i\lambda-\beta|^2} = \int_{\mathbb{R}^n} \frac{\lambda_j U(d\lambda)\tilde{\chi}_k}{|i\lambda-\beta|^2}$$

which implies

$$D_k\tilde{\phi}^\beta - \beta\tilde{\phi}^\beta = \int_{\mathbb{R}^n} \sum_j \frac{(-i\lambda_j-\beta)(i\lambda_k-\beta)}{|i\lambda-\beta|^2} U(d\lambda)\tilde{\chi}_j(\omega)$$

$$= \int_{\mathbb{R}^n} \sum_j \frac{(-i\lambda_j-\beta)(i\lambda_j-\beta)}{|i\lambda-\beta|^2} U(d\lambda)\tilde{\chi}_k(\omega) = \tilde{\chi}_k.$$

Therefore, (4.62) reads

$$E\tilde{a}_{ij}\tilde{\chi}_j\tilde{\chi}_i + \beta E\tilde{a}_{ij}\tilde{\chi}_j\tilde{\phi}^\beta = 0. \tag{4.63}$$

However,

$$E|\beta\tilde{\phi}^\beta|^2 = \int_{\mathbb{R}^n} \beta^2 \sum_{j,k} \frac{(-i\lambda_j-\beta)(-i\lambda_k-\beta))}{|i\lambda-\beta|^4} EU(d\lambda)\tilde{\chi}_j\tilde{\chi}_k$$

$$\to \sum_{j,k} EU\{0\}\tilde{\chi}_j\tilde{\chi}_k \text{ as } \beta \to 0$$

and by ergodicity and property (4.60), we have

$$E|\beta\tilde{\phi}^\beta|^2 \to 0 \quad \text{as } \beta \to 0.$$

Therefore, (4.63) implies

$$E\tilde{a}_{ij}\tilde{\chi}_j\tilde{\chi}_i = 0$$

hence $\tilde{\chi}_i = 0$, which implies also $\lambda = 0$.

Existence:

Let $\beta > 0$, we solve the problem

$$-\frac{\partial}{\partial y}(a_{ij}\frac{\partial \chi^\beta}{\partial y_0}) + \beta\chi^\beta = \frac{\partial g_j}{\partial y_j}.$$
$$\chi^\beta(y;\omega) = T_y\tilde{\chi}^\beta, \tilde{\chi}^\beta \in \mathcal{H}^1 \tag{4.64}$$

This problem is equivalent to

$$E\tilde{a}_{iy}D_j\tilde{\chi}^\beta D_i\tilde{\phi} + \beta E\tilde{\chi}^\beta\tilde{\phi} = -E\tilde{g}_j D_j\tilde{\phi}, \quad \forall\tilde{\phi} \in \mathcal{H}^1 \tag{4.65}$$

We easily deduce the estimates

$$E|D_j\tilde{\chi}^\beta|^2 \leq C$$

$$\beta E(\tilde{\chi}^\beta)^2 \leq C.$$

Let us extract a subsequence such that

$$D_j\tilde{\chi}^\beta \rightharpoonup \tilde{\chi}_j \text{ in } H \text{ weakly.}$$

$$ED_j\tilde{\chi}^\beta D_k\tilde{\phi} = ED_k\tilde{\chi}^\beta D_j\tilde{\phi}$$

$$ED_j\tilde{\chi}^\beta = 0$$

we deduce (4.59) and (4.60). Going to the limit in (4.65) we have

$$E\tilde{a}_{ij}\tilde{\chi}_j D_j\tilde{\phi} = -E\tilde{g}_j D_j\tilde{\phi} \quad \forall\tilde{\phi} \in \mathcal{H}^1. \tag{4.66}$$

Define then

$$\chi(y;\omega) = \int_{\mathbb{R}^n} (e^{i\lambda y} - 1)\frac{1}{|\lambda|^2}\sum_j (-i\lambda_j) U(d\lambda)\tilde{\chi}_j(\omega) \tag{4.67}$$

then

$$\begin{aligned}\frac{\partial \chi}{\partial y_k}(y;\omega) &= \int_{\mathbb{R}^n} e^{i\lambda y}\frac{1}{|\lambda|^2}\sum_j \lambda_j\lambda_k U(d\lambda)\tilde{\chi}_j(\omega) \\ &= T_y\tilde{\chi}_k\end{aligned}$$

and $\chi(0;\omega) = 0$, $E\chi(y) = 0$. Then (4.66) can be written as

$$-D_i\tilde{a}_{ij}\tilde{\chi}_j = D_i\tilde{g}_i \text{ equality in } \mathcal{H}^{-1}$$

which is indeed (4.57).

In (Bensoussan and Blankenship 1988) this result was used to study the homgenization of nonlinear, stochastic control problems. In (Begis and Blankenship 1987) numerical methods for the evaluation of homogenized representations to certain problems in the mechanics of period composite materials were discussed (see also (Begis, Duvaut, and Hassim 1981)). The problem of designing numerical methods for the treatment of random media remains open.

5 A Macroscopic Scattering Model

In this section we describe a computational model for the scattering and absorption of EM radiation by a volume of foliage covered terrain. In effect we combine the detailed models of the previous sections with a model for the foliage as a "very rough interface" between the air and the underlying earth. The effective parameter models developed in the earlier sections are used to characterize the scattering and absorption of the EM radiation within the microstructure of the foliage. The result is an effective parameter representation for the scattering volume. A rough surface (interface) scattering model is then used to characterize the interaction of the radiation with the "layer" of foliage. We use the method of smoothing perturbation (Keller 1964) to treat this "macroscopic" aspect of the interaction.

Typical regions of foliage covered terrain, viewed as a scattering surface, are *very rough* in the sense that they do not satisfy the validity conditions of the classical Kirchhoff integral method and the Rayleigh-Rice perturbation methods. In particular, scattering regimes in which the rms value h of the surface height variations is substantially larger than their correlation length ℓ, while λ, the wavelength of the incident radiation is comparable to h ("the low frequency case") or to ℓ ("the high frequency case") are incompatible with the classical methods. We shall call these surfaces "very rough." The local angles of incidence between very rough surfaces and the incoming radiation are not always small, even in operational geometries which suggest near "grazing" incidence. Hence, it is necessary to have methods suitable for all angles of incidence.

Our solution technique, which is an approximate method, involves several steps. First, the foliage covered terrain is regarded as a rough surface, which is in turn regarded as constituting a "layer" of inhomogeneous material separating two infinite slabs of (essentially) homogeneous material (e.g., air and earth). Using this paradigm, the problem is reformulated as scattering in a random medium in which the inhomogeneity is characterized by a small, dimensionless number $\nu = \lambda/h$.[9]

The method of smoothing perturbation is applied to the vector wave equation for the electric field vector in the entire space in a manner similar to the work of Keller and Karal (1964). This leads to a deterministic wave equation for the average field which involves the second order statistics - the correlation function - of the surface fluctuations. Solution of this equation may be readily carried out using *reflection and transmission coefficients* for the random layer, and these parameters constitute an effective, simplified representation of the original scattering problem which requires a minimum of empirical data for implementation. In numerical studies we have used the technique to treat surfaces with uniform and Gaussian distributed height variations.[10] The reflection coefficients differ only slightly from the Fresnel coefficients at grazing incidence; however, at larger incident angles substantial differences (20-50%) in both amplitude and phase variations are obtained.

[9]This parameter is in addition to the small parameters characterizing mean free path length in the random medium models used in the previous sections.

[10]See (Blankenship 1986) and (Barakati 1983). A sample of these results is included below.

5.1 The Method of Smoothing Perturbation

The mathematical technique we use is a standard method in perturbation theory which has a simple, general formulation (Keller 1964). Let u_0 be a wave in a homogeneous medium characterized by a linear operator L_0, so $L_0 u_0 = 0$, with suitable boundary conditions. Let (Ω, F, P) be a probability space, and let $\omega \in \Omega$ designate a different medium characterized by the operator $L_0 - \nu L_1(\omega) - \nu^2 L_2(\omega) + O(\nu^3)$. Here ν is a measure of the inhomogeneity of the medium, and L_1 and L_2 are perturbation operators representing the effects of inhomogeneity. A (random) wave in this medium satisfies

$$[L_0 - \nu L_1(\omega) - \nu^2 L_2(\omega) + O(\nu^3)]u^\nu(\omega) = 0. \tag{5.1}$$

Let $< u >$ be the expected value of $u(\omega)$ as a random variable on (Ω, F, P),

$$< u >= \int_\Omega u(\omega) P(d\omega).$$

Our objective is to find an equation for $< u^\nu >$ accurate to $O(\nu^3)$. When $< L_1 >= 0$, we can write

$$[L_0 - \nu^2(< L_1 L_0^{-1} L_1 > + < L_2 >)] < u^\nu >= O(\nu^3) \tag{5.2}$$

When the $O(\nu^3)$ terms are omitted, this is an explicit, deterministic equation for $< u^\nu >$ which is accurate to $O(\nu^3)$ (see the remarks on convergence below). This equation may be treated by a number of methods.

In many applications $u^\nu(x, \omega)$ is a vector function of the (vector) position x. Then L_0, L_1, and L_2 are matrix operators, each component of which may be a differential operator. In these cases L_0^{-1} is an integral operator, the kernel of which is the (dyadic)

Green's function $G_0(x,y)$, a matrix function defined by

$$L_0 G_0(x,y) = I\delta(x-y) \tag{5.3}$$

where I is the identity matrix and δ is the Dirac delta function. Then, in terms of the Green's function we have

$$L_0(x) < u^\nu(x) > -\nu^2 < L_1(x) \int G_0(x,y) L_1(y) < u^\nu(y) > dy >= O(\nu^3) \tag{5.4}$$

It is clear from this how the second order statistics - cross correlations and autocorrelations - enter the expression for $< u^\nu >$.

Notice that the smoothing perturbation leads to an integral equation for the approximation of $< u >$. This equation is more complex than the differential equations which arise in other approximation methods. This complexity leads to greater accuracy and to an increased range of validity in the approximation. The papers (Chow 1974) (Blankenship 1979) contain a comparative analysis of several approximations, including the smoothing perturbation.

5.2 Random Surface Model

Consider the physics of a plane, time-harmonic EM wave of arbitrary polarization incident on a very rough interface separating air and a lossy dielectric (foliage). The rough interface is represented by

$$z = g^\nu(r,\omega) = g(\frac{r}{\nu},\omega) \tag{5.5}$$

$$r = (x,y)$$

where $g(r,\omega)$ is a spatially homogeneous random field, and $\omega \in \Omega$, (Ω, F, P) a probability space, is a realization of that random field. We assume that $|g(r,\omega)| \leq h/2$ for some $h > 0$ and $\forall r$. The dimensionless parameter $\nu > 0$ is the correlation length of $g(r,\omega)$ normalized by h . We are interested in the surface scattering problem as $\nu \to 0$. We shall assume that the random surface has zero mean

$$< g(r,\cdot) >= 0. \tag{5.6}$$

This puts us in the framework corresponding to equations (5.2,5.4). Spatial homogeneity means that

$$< g(r,\cdot)g(s,\cdot) >= R(r-s). \tag{5.7}$$

And if

$$\ell \triangleq [\int_{R^2} R(r)dr/R(0)]^{1/2}$$

is the correlation length of the surface variations (recall $h = R(0)$), then $\nu = \ell/h$. The example

$$\begin{aligned} R(r) &= h^2 e^{(-|r|/\ell)} \\ &= h^2 e^{(-|\hat{r}|/\nu)} \end{aligned} \tag{5.8}$$

with $\hat{r} = r/h$ illustrates the definition.

The surface cross section shown in Figure 5.2 illustrates the scattering geometry and the parameters. For a fixed h, as $\nu \to 0$, the surface fluctuates more rapidly about the plane $z = 0$, and the incident radiation encounters an increasingly rugged interface. As the figure suggests, the surface variations are confined to a "layer" of thickness h. We shall use this artificially defined layer as the basis for our analysis.

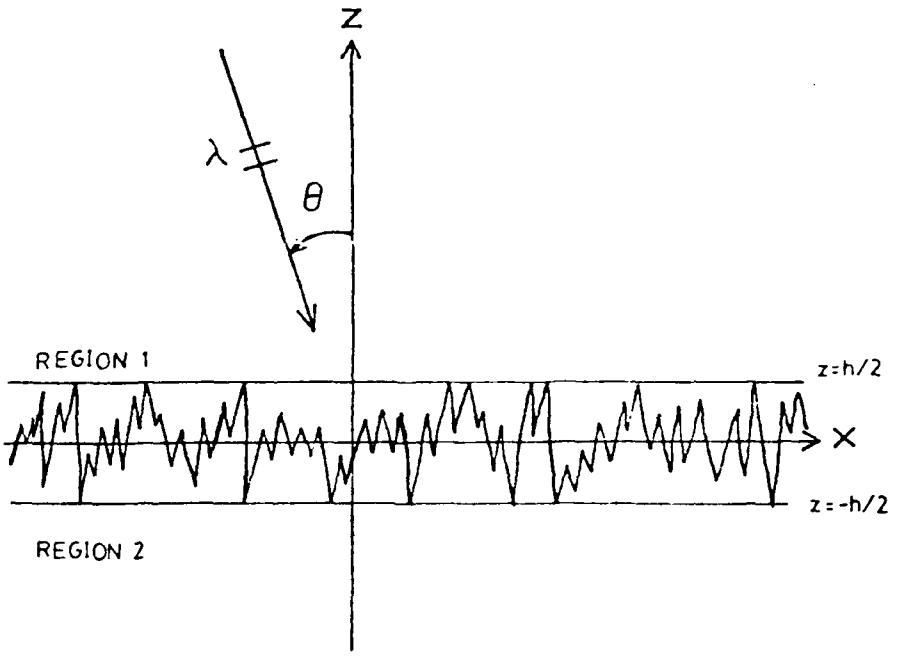

Figure 5.1: Rough surface scattering.

5.3 Vector Wave Equation and the Smoothing Approximation

Starting with Maxwell's equation for time–harmonic fields, we can obtain the following vector wave equation for the electric field vector E

$$\nabla \times \nabla \times E^{\nu} - k^2 N_{\nu}^2(r,\omega) E^{\nu} = 0 \tag{5.9}$$

valid in the entire space $r = (x, y, z)$. Here $k = 2\pi/\lambda$ is the propagation constant in free space, and $N_{\nu}^2(r,\omega)$ is the random dielectric constant associated with the layered structure defined in Figure 5.2. We have assumed that the magnetic permeability μ is uniform ($\mu = 1$) throughout space. Defining the function

$$H^{\nu}(r,\omega) = \begin{cases} 1 & z \geq g^{\nu}(r,\omega) \\ 0 & \text{elsewhere} \end{cases} \tag{5.10}$$

then

$$N_\nu^2(r,\omega) = H^\nu(r,\omega) + \eta^2[1 - H^\nu(r,\omega)] \tag{5.11}$$

where $\eta^2 = \epsilon_r + j60\lambda\sigma$, σ the conductivity, is the complex dielectric constant of the material in region 2 of Figure 5.2.[11]

The single (vector) equation (5.9) together with a radiation boundary condition for $E^\nu(r)$ as $|r| \to \infty$ is a reformulation of the EM interface problem which includes a wave equation for $z > g^\nu(r,\omega)$, a wave equation for $z < g^\nu(r,\omega)$, and the usual continuity relation for the fields in regions 1 and 2 across the interface. Thus, (5.9)-(5.11) is a reformulation of the Maxwell's equation with a random boundary condition as a partial differential equation with a random coefficient.[12] This formulation enables us to apply the smoothing perturbation to (5.9). Let

$$H_0(z) = < H^\eta(r,\cdot) >$$

$$N_0^2(z) = H_0(z) + \eta^2[1 - H_0(z)] \tag{5.12}$$

$$\tilde{H}^\nu(r,\omega) = [H^\nu(r,\omega) - H_0(z)]/\nu$$

(recall the assumption of spatial homogeneity). Then (5.9) may be rewritten as

$$\nabla \times \nabla \times E^\nu - k^2 N_0^2(z) E^\nu - \nu^2 k^2 (1 - \eta^2) \tilde{H}^\nu(r,\omega) E^\nu = 0. \tag{5.13}$$

[11]The complex dielectric constant would be computed or estimated using the methods discussed in the previous sections.

[12]Note the similarity to the models used in the section on homogenization. In effect, the models here are "large scale" representations of the scattering phenomena; whereas those used in the homogenization technique are local, "small scale" representations. It would be possible to combine the rough surface paradigm with the model used to represent microscopic events; however, the resulting analysis would be rather complex.

Identifying

$$L_0 = \nabla \times \nabla - k^2 N_0^2(z)$$

$$L_1 = -k^2(1-\eta^2)\tilde{H}^\nu(r,\omega)$$

$$L_2 = 0$$

and noting $< L_1 >= 0$, we can apply the smoothing perturbation to (5.13). That is, $< E^\nu >$ satisfies, formally to $O(\nu^3)$,

$$L_0 < E^\nu > -\nu^2 < L_1 L_0^{-1} L_1 >< E^\nu >= 0 \tag{5.15}$$

Let $G_0(r|\hat{r})$ be the dyadic Green's function of the operator L_0 with an outgoing wave condition (Tai 1971); i.e.,

$$L_0 G_0(r|\hat{r}) = I\delta(r|\hat{r}) \tag{5.16}$$

We can then write (5.15) as

$$L_0 < E^\nu > (r) - \nu^2 k^4 (1-\eta^2)^2 \int\int_{\Re^3}\int G_0(r|\hat{r})$$

$$\cdot R\left(\frac{x-\hat{x}}{\nu}, \frac{y-\hat{y}}{\nu}, z, \hat{z}\right) < E^\nu > (r)\frac{d\hat{x}}{\nu}\frac{d\hat{y}}{\nu}d\hat{z} = 0 \tag{5.17}$$

where

$$R\left(\frac{x-\hat{x}}{\nu}, \frac{y-\hat{y}}{\nu}, z, \hat{z}\right) =< \tilde{H}^\nu(r,\cdot)\tilde{H}^\nu(\hat{r},\cdot) > \tag{5.18}$$

Setting $\tilde{x} = (x-\hat{x})/\nu, \tilde{y} = (y-\hat{y})/\nu, \tilde{z} = \hat{z}$, we can rewrite (5.17) as

$$L_0 < E^\nu > (r) - \nu^2 k^4 (1-\eta^2)^2 \int\int_{\Re^3}\int G_0(\nu\tilde{x}, \nu\tilde{y}, z, \tilde{z}) \tag{5.19}$$

$$\cdot R(\tilde{x}, \tilde{y}, z, \tilde{z}) < E^\nu > (x-\nu\tilde{x}, y-\nu\tilde{y}, \tilde{z})d\tilde{x}d\tilde{y}d\tilde{z} = 0$$

Now assume that R is integrable over $(\tilde{x}, \tilde{y}) \in \Re^2$. (See Appendix I for comments on this assumption.) Then for $< E^\nu >$ bounded, the integral in (5.19) will be bounded since $R(\tilde{x}, \tilde{y}, z, \tilde{z}) = 0$ for $|z|, |\tilde{z}| \leq h/2$.

Adopting the ansatz

$$< E^\nu > = < E >_0 + \nu^2 < E >_2 + \nu^4 < E >_4 + \ldots \tag{5.20}$$

and substituting in (5.19), we find

$$L_0 < E >_0 = 0 \tag{5.21}$$

$$L_0 < E >_2 = k^4(1-\eta^2)^2 \int_{-h/2}^{h/2} G_0(0,0,z,\tilde{z}) \int\int_{\Re^2} R(\tilde{x},\tilde{y},z,\tilde{z}) d\tilde{x} d\tilde{y} \cdot < E >_0 (x,y,z,\tilde{z}) d\tilde{z} \tag{5.22}$$

Thus, $< E >_0$ approximates the average electric field, formally, to $O(\nu^2)$. An $O(\nu^3)$ approximation may be obtained by substituting $< E >_0$ into (5.22) and solving for $< E >_2$. We shall not consider this extension here.

5.4 Reflection Coefficients

To illustrate the computations involved and the qualitative effects of the surface roughness on reflectivity, we shall compute the reflection coefficients for a surface whose deviations from the mean plane are uniformly distributed in $[-h/2, h/2]$. This means that $H_0(z)$ is linear in $[-h/2, h/2]$; and, in effect, it presumes minimum information about the detailed structure of the rough surface. We shall treat horizontally and vertically polarized incident waves separately. Arbitrary polarized incident waves may be treated by superposition.

Let the waves be incident in the $x-z$ plane and let θ be the angle of incidence of the (plane) wave. (See Figure 5.4.) Let $\hat{h}_i$ and $\hat{v}_i$ denote the unit vectors in the direction of the incident electric field vector for horizontal and vertical polarization, respectively. Similarly, $\hat{h}_r$ and $\hat{v}_r$ are the corresponding unit vectors for the reflected field. Let $\hat{k}_i$ and $\hat{k}_r$ denote the unit vectors along the directions of propagation of the incident and reflected waves, respectively. Using $\hat{x}, \hat{y}$ and $\hat{z}$ to denote the unit coordinate vectors, we can define the polarization by

$$\hat{h}_i = \hat{h}_r = \hat{x} \times \hat{z} = -\hat{y}$$

$$\hat{v}_i = \hat{k} \times \hat{k}_i, \quad \hat{v}_r = \hat{k}_r \times \hat{h}_r \tag{5.23}$$

In the case being considered

$$\hat{k}_i = \sin\theta\hat{x} - \cos\theta\hat{z}, \quad \hat{k}_r = \sin\theta\hat{x} + \cos\theta\hat{z} \tag{5.24}$$

and this in (5.23) means that

$$\hat{v}_i = -\cos\theta\hat{x} - \sin\theta\hat{z}, \quad \hat{v}_r = \cos\theta\hat{x} - \sin\theta\hat{z} \tag{5.25}$$

5.4.1 Horizontal Polarization

In this case the average electric field vector in $z \geq h/2$ can be written as a combination of the incident and reflected waves as

$$< E > = e^{jk'x\sin\theta - (z - h/2)\cos\theta]}\hat{h}_i \tag{5.26}$$

$$+(\rho_{HH}\hat{h}_r + \rho_{HV}\hat{v}_r)e^{jk'x\sin\theta + (z - h/2)\cos\theta]}, z \geq h/2$$

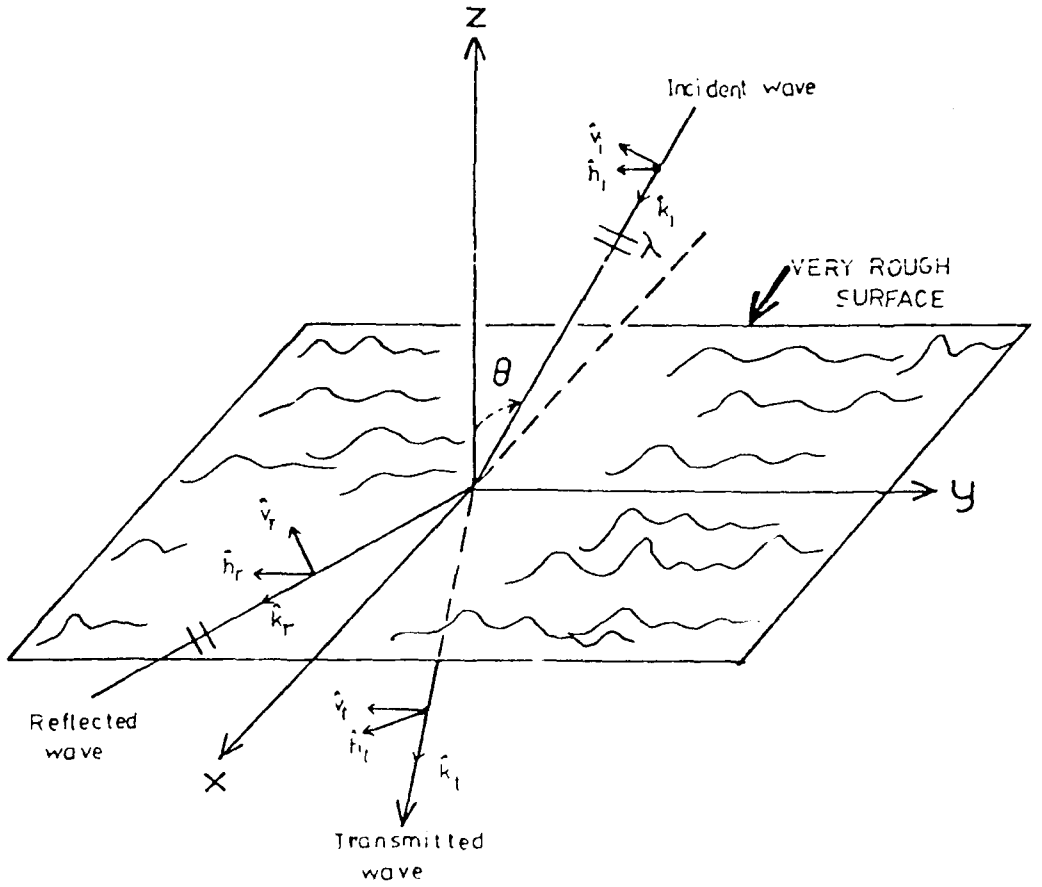

Figure 5.2: Scattering geometry.

where ρ_{HH} and ρ_{HV} are the reflection coefficients associated with horizontally polarized incident radiation. The coefficient ρ_{HV} will be zero if there is no depolarization due to scattering. (This is the case, at least to order zero in ν.)

Having assumed uniformly distributed surface deviations, the vector wave equation (5.21) for the zeroth order average scattered field is

$$\nabla \times \nabla \times < E >_0 - k^2 f(z) < E >_0 = 0 \tag{5.27}$$

where

$$f(z) = \begin{cases} 1 & z \geq h/2 \\ \frac{1-\eta^2}{h} z + \frac{1+\eta^2}{2} & |z| \leq h/2 \\ \eta^2 & z < -h/2 \end{cases} \tag{5.28}$$

From this we see that (5.26) is, in fact, a solution of (5.27,5.28).

The field in $z < -h/2$ can be written as

$$< E >_0 = (\gamma_{HH} \hat{h}_t + \gamma_{HV} \hat{v}_t) e^{jk[x \cos\theta - (z+h/2)\sqrt{\eta^2 - \sin^2\theta}]}, \quad z \leq -h/2 \tag{5.29}$$

where we have required $< E >_0$ to be outgoing and finite as $z \to -\infty$. The unit vectors $\hat{h}_t$ and $\hat{v}_t$ denote the transmission directions of the horizontally and vertically polarized waves, respectively. They are defined by

$$\hat{h}_t \qquad \hat{z} = -\hat{y}$$

$$\bar{k}_t = k \sin\theta \hat{x} - k\sqrt{\eta^2 - \sin^2\theta}\,\hat{z} \tag{5.30}$$

$$\hat{v}_t = \frac{\bar{k}_t \times \hat{h}_t}{|\bar{k}_t|} = \frac{\sqrt{\eta^2 - \sin^2\theta}}{\eta} \hat{x} - \frac{\sin\theta}{\eta} \hat{z}$$

The constants γ_{HH} and γ_{HV} are transmission coefficients.

The electric field in $|z| \leq h/2$ can be written as a linear combination of two vector wave functions, M and N (Tai 1971)

$$M = \nabla \times (\psi \hat{z}) \tag{5.31}$$

$$N = \frac{1}{kf(z)} \nabla \times \nabla \times (\phi \hat{z})$$

which satisfy (5.27,5.28) if we take the scalar functions ψ and ϕ as solutions of

$$\nabla^2 \psi + k^2 f(z) \psi = 0 \tag{5.32}$$

$$\nabla^2 \phi - \frac{1}{f(z)} \frac{df(z)}{dz} \frac{\partial \phi}{\partial z} + k^2 f(z) \phi = 0$$

Assuming a separation of variables

$$\psi = X_1(x) Y_1(y) Z_1(z) \tag{5.33}$$

$$\phi = X_2(x) Y_2(y) Z_2(z)$$

and separation constants k^2a^2 and k^2b^2, we obtain from (5.32)

$$(\frac{d^2}{dx^2} + k^2 a^2) X_i(x) = 0, i = 1, 2 \tag{5.34}$$

$$(\frac{d^2}{dy^2} + k^2 b^2) Y_i(y) = 0, i = 1, 2 \tag{5.35}$$

$$[\frac{d^2}{dz^2} + k^2 (f(z) - a^2 - b^2)] Z_1(z) = 0 \tag{5.36}$$

$$[\frac{d^2}{dz^2} - \frac{1}{f(z)} \frac{df(z)}{dz} \frac{d}{dz} + k^2 (f(z) - a^2 - b^2)] Z_2(z) = 0 \tag{5.37}$$

The form of the incident wave suggests

$$a = \sin\theta, \quad b = 0 \tag{5.38}$$

so that

$$X_i(x) = e^{jkx\sin\theta}, \quad i = 1,2$$

$$Y_i(y) = 1, \quad i = 1,2$$

The general solution of (5.36) may be written using special functions as

$$Z_1(z) = w^{1/3}[AH^{(1)}_{1/3}(w) + BH^{(2)}_{1/3}(w)] \tag{5.40}$$

where

$$w = w(z) = \frac{2}{3c}(cz + d)^{3/2} \tag{5.41}$$

$$c = k^2\frac{(1-\eta^2)}{h}, \quad d = \frac{1}{2}k^2(1+\eta^2) - k^2\sin^2\theta$$

and $H^{(i)}_{\mu}(\cdot), i = 1,2$, are the Hankel functions of the i-th kind of order μ (Abramowitz and Stegun 1971).

The second order equation (5.37) with $f(z)$ linear cannot be reduced to a standard form. We can, however, construct two linearly independent, series solutions $Z_{21}(z_1)$ and $Z_{22}(z_1)$ in terms of $z_1 = f(z), |z| \leq h/2$; see Appendix II for details. Note that the values of Z_{21} and Z_{22} are needed at only two points, which makes the numerical computation particularly attractive.

Writing a general solution of (5.37) as

$$Z_2 = CZ_{21}(z_1) + DZ_{22}(z_1) \tag{5.42}$$

we can write a general expression for the average electric field in the layer $|z| \leq h/2$

$$< E_{x0} > = [j\sin\theta\frac{(1-\eta^2)}{h}\frac{C\hat{z}_{21}(z_1) + D\hat{z}_{22}(z_1)}{f(z)}\hat{x}]$$

$$- [jk \sin\theta w^{1/3}(AH^{(1)}_{1/3}(w) + BH^{(2)}_{1/3}(w))\hat{y}] \tag{5.43}$$

$$+ [k \sin^2\theta (CZ_{21}(z_1) + DZ_{22}(z_1))\hat{z}], |z| \leq h/2$$

where A, B, C, and D are complex constants to be determined from the boundary conditions at $= \pm h/2$.

The boundary conditions satisfied by $< E >_0$ are derived from the requirements that the tangential components of the electric and magnetic fields must be continuous across the (artificial) "boundaries" of the "layer" at $z = h/2$. These conditions reduce to the requirements:

(i) $z \times < E >_0$ is continuous at $z = h/2$ and $z = -h/2$

(ii) $z \times \nabla \times < E >_0$ is continuous at $z = h/2$ and $z = -h/2$.

Each of these conditions yields four equations, so we have eight linear equations relating the eight unknowns $\rho_{HH}, \rho_{HV}, A, B, C, D, \gamma_{HH}$, and γ_{HV}. These equations decouple into two sets of equations, and they may be written as

$$\begin{bmatrix} -1 & a_{12} & a_{13} & 0 \\ 0 & a_{22} & a_{23} & -1 \\ -jk\cos\theta & a_{32} & a_{33} & 0 \\ 0 & a_{42} & a_{43} & jk\sqrt{\eta^2 - \sin^2\theta} \end{bmatrix} \begin{bmatrix} \rho_{HH} \\ A \\ B \\ \gamma_{HH} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -jk\cos\theta \\ 0 \end{bmatrix} \tag{5.44}$$

where

$$a_{12} = w_+^{1/3} H^{(1)}_{1/3}(w_+), \quad a_{13} = w_+^{1/3} H^{(2)}_{1/3}(w_+)$$

$$a_{22} = w_-^{1/3} H^{(1)}_{1/3}(w_-), \quad a_{23} = w_-^{1/3} H^{(2)}_{1/3}(w_-)$$

$$a_{32} = (d + ch/2)^{1/2} w_+^{1/3} H_{-2/3}^{(1)}(w_+), \quad a_{33} = (d + ch/2)^{1/2} w_+^{1/3} H_{-2/3}^{(2)}(w_+)$$

$$a_{42} = (d - ch/2)^{1/2} w_-^{1/3} H_{-2/3}^{(1)}(w_-), \quad a_{43} = (d - ch/2)^{1/2} w_-^{1/3} H_{-2/3}^{(2)}(w_-)$$

and

$$\begin{bmatrix} -1 & b_{12} & b_{13} & 0 \\ 0 & b_{22} & b_{23} & -1 \\ -ik\cos\theta & b_{32} & b_{33} & 0 \\ 0 & b_{42} & b_{43} & jk\sqrt{\eta^2 - \sin^2\theta} \end{bmatrix} \begin{bmatrix} \rho_{HV} \\ C \\ D \\ \gamma_{HV} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{5.45}$$

where

$$b_{12} = Z_{21}(1), \quad b_{13} = Z_{22}(1)$$

$$b_{22} = \frac{1}{\eta} Z_{21}(\eta^2), \quad b_{23} = \frac{1}{\eta} Z_{22}(\eta^2)$$

$$b_{32} = \frac{(1-\eta^2)}{h} \hat{z}_{21}(1), \quad b_{33} = \frac{(1-\eta^2)}{h} \hat{z}_{22}(1)$$

$$b_{42} = \frac{(1-\eta^2)}{h} \hat{z}_{21}(\eta^2), \quad b_{43} = \frac{(1-\eta^2)}{h} \hat{z}_{22}(\eta^2)$$

and $w_\pm = w(z = \pm h/2)$. In writing (5.44,5.45) we have replaced A and B by $jkA\sin\theta$ and $jkB\cos\theta$, respectively. We have also absorbed a factor, $k\sin\theta$, into C and D and used the fact (Abramowitz and Stegun 1971) that

$$\frac{d}{dw}(w^{1/3} H_{1/3}^{(i)}(w)) = w^{1/3} H_{-2/3}^{(i)}(w), \quad i = 1, 2 \tag{5.46}$$

and $dw/dz = (cz + d)^2$. The roots of complex quantities are interpreted as principal values. For convenience we write (5.44,5.45) in compact notation as

$$P_H q_{HH} = s \tag{5.47}$$

$$P_V q_{HV} = 0 \tag{5.48}$$

with the definitions of P_H, P_V, etc. obvious from (5.44,5.45).

Numerical results show the matrix P_V in (5.45) to be nonsingular. So we obtain

$$\rho_{HV} = C = D = \gamma_{HV} = 0 \tag{5.49}$$

for horizontal polarization. This means that, to zero order in the small parameter ν, there is no change in polarization in scattering or transmission. The reflection coefficient ρ_{HH} is computed from (5.44).

5.4.2 Vertical Polarization

The field in $z \geq h/2$ may be written

$$< E >_0 = e^{jk[x\sin\theta - (z-h/2)\cos\theta]}\hat{v}_i \tag{5.50}$$

$$+(\rho_{HV}\hat{h}_r + \rho_{VV}\hat{v}_r)e^{jk[x\sin\theta + (z-h/2)\cos\theta]}, \quad z \geq h/2$$

The expression for the field in $|z| \leq h/2$ is unchanged from (5.43); and in $z < -h/2$ we have

$$< E >_0 = (\gamma_{VH}\hat{h}_t + \gamma_{VV}\hat{v}_t)e^{jk[x\sin\theta - (z+h/2)\sqrt{\eta^2 - \sin^2\theta}]}, \quad z \leq -h/2. \tag{5.51}$$

Applying the boundary conditions at $z = \pm\frac{h}{2}$, we obtain

$$P_H q_{VH} = 0 \tag{5.52}$$

$$P_V q_{VV} = s \tag{5.53}$$

where

$$q_{VH} = \begin{bmatrix} \rho_{VH} \\ A \\ B \\ \gamma_{VH} \end{bmatrix}, \quad q_{VV} = \begin{bmatrix} \rho_{VV} \\ C \\ D \\ \gamma_{VV} \end{bmatrix} \tag{5.54}$$

with P_H, P_V, and s as in (5.44,5.45,5.47,5.48). Once again numerical computations show that there is no depolarization to zero order in the small parameter $\nu = l/h$; that is, $\rho_{VH} = A = B = \gamma_{VH} = 0$. The reflection coefficients may be computed from the 4×4 system (5.53).

5.5 Typical Numerical Results

Figures 5.5-5.5 show the magnitude and phase of ρ_{HH} and ρ_{VV} computed from (5.47) and (5.53), respectively, for a very rough interface with $\epsilon_r = 80, \sigma = 4.3 mho/m, \lambda = 2.5cm$ and $h = \lambda/2, \lambda, 2\lambda$. For comparison we have included the magnitudes and phase of the complex Fresnel reflection coefficients for horizontal and vertical polarization (Northam 1981).

$$\rho_{FH} = \frac{\cos\theta - \sqrt{\eta^2 - \sin^2\theta}}{\cos\theta + \sqrt{\eta^2 - \sin^2\theta}} \tag{5.55}$$

$$\rho_{FV} = \frac{\eta^2\cos\theta - \sqrt{\eta^2 - \sin^2\theta}}{\eta^2\cos\theta + \sqrt{\eta^2 - \sin^2\theta}}$$

The plots show the coefficients against the *grazing angle*, $90^\circ - \theta$, rather than the angle of incidence θ.

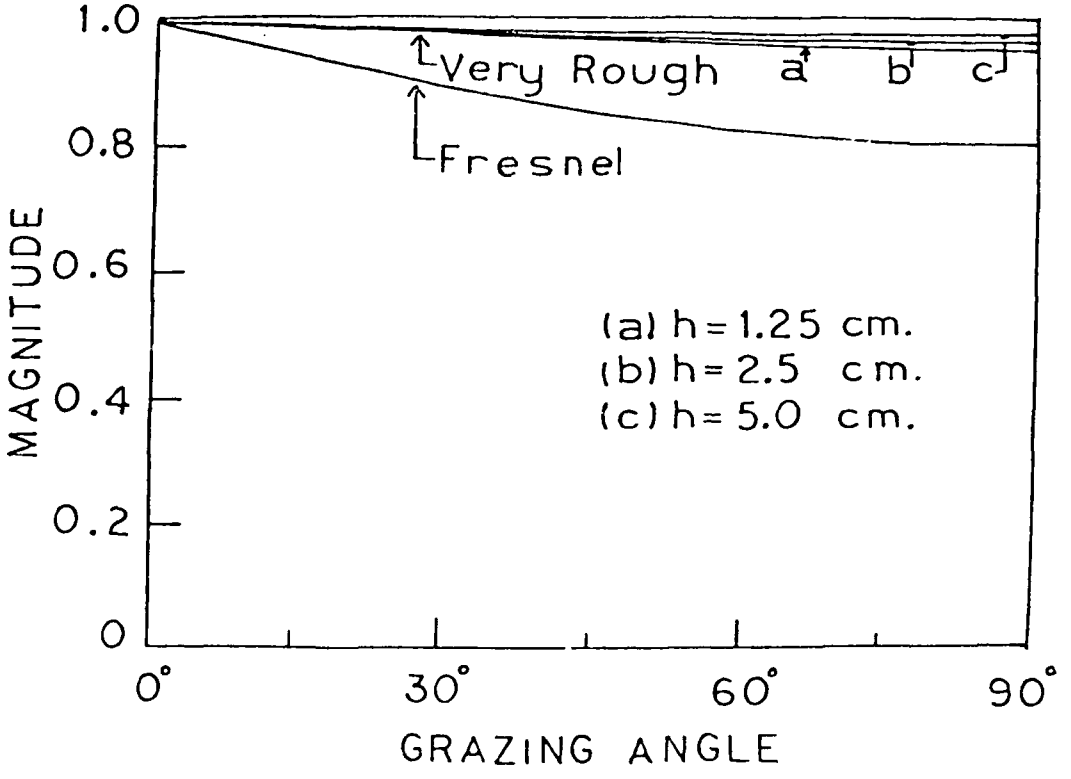

Figure 5.3: Magnitude of ρ_{HH} versus grazing angle and h.

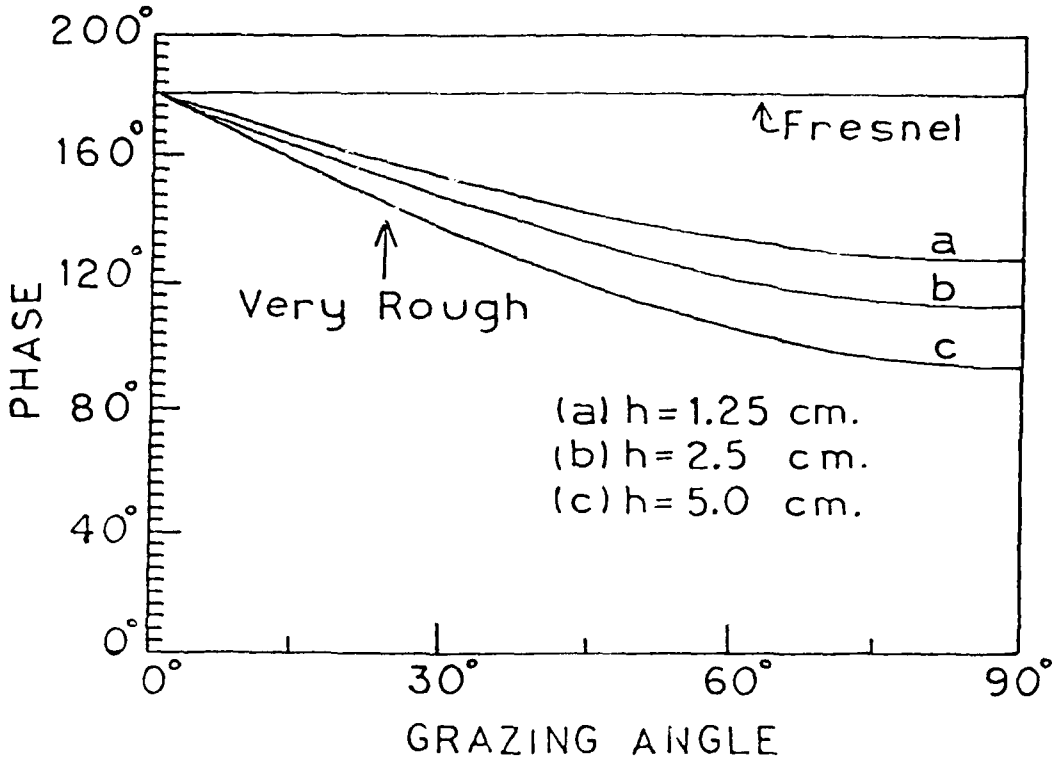

Figure 5.4: Phase of ρ_{HH} versus grazing angle and h

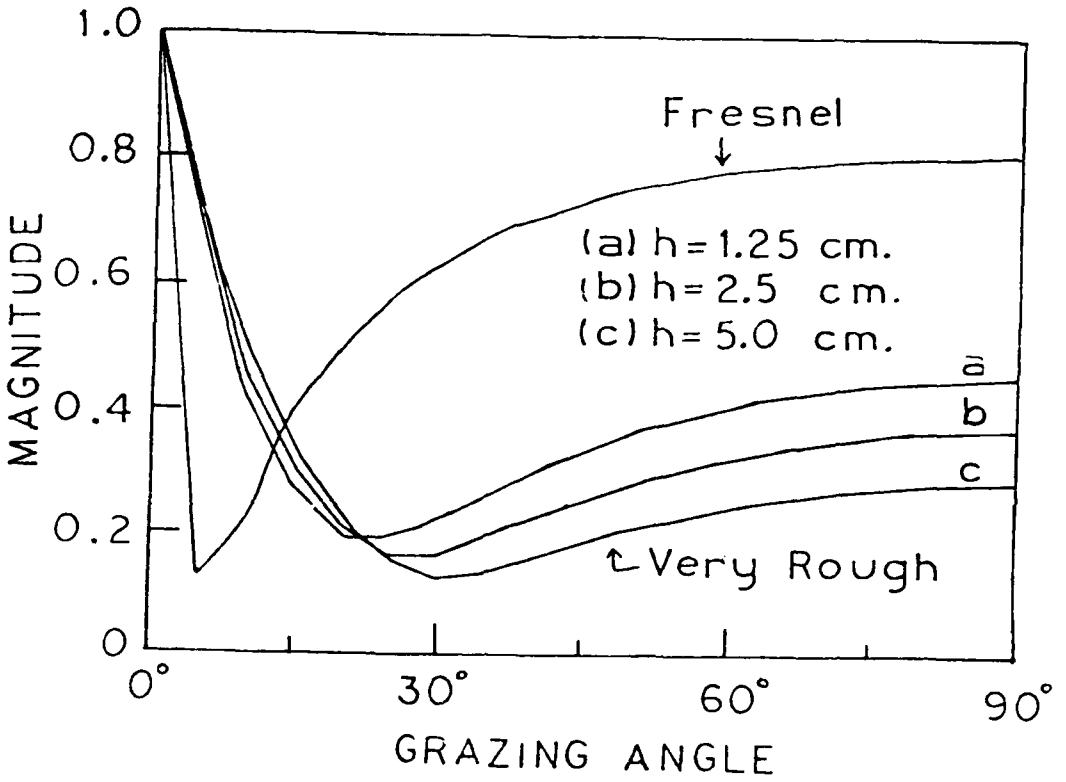

Figure 5.5: Magnitude of ρ_{VV} versus grazing angle and h.

Evidently, the differences between the reflection coefficients of a very rough surface and the Fresnel coefficients of a flat surface become pronounced as the grazing angle increases. At normal incidence the magnitude of the reflection coefficient $|\rho_{HH}|$ of the very rough surface is about 20% larger than that, $|\rho_F H|$, of a flat surface. The phase difference, Figure 5.5, is even more pronounced. And the differences are exaggerated even further in the case of vertical polarization. We have shown the dependence of ρ_{HH} and ρ_{VV} on h for $h = \lambda/2, \lambda, 2\lambda$. We find that as h decreases the magnitude of the reflection coefficient decreases and the phase increases. The Fresnel coefficients define the limits as $h \to 0$.

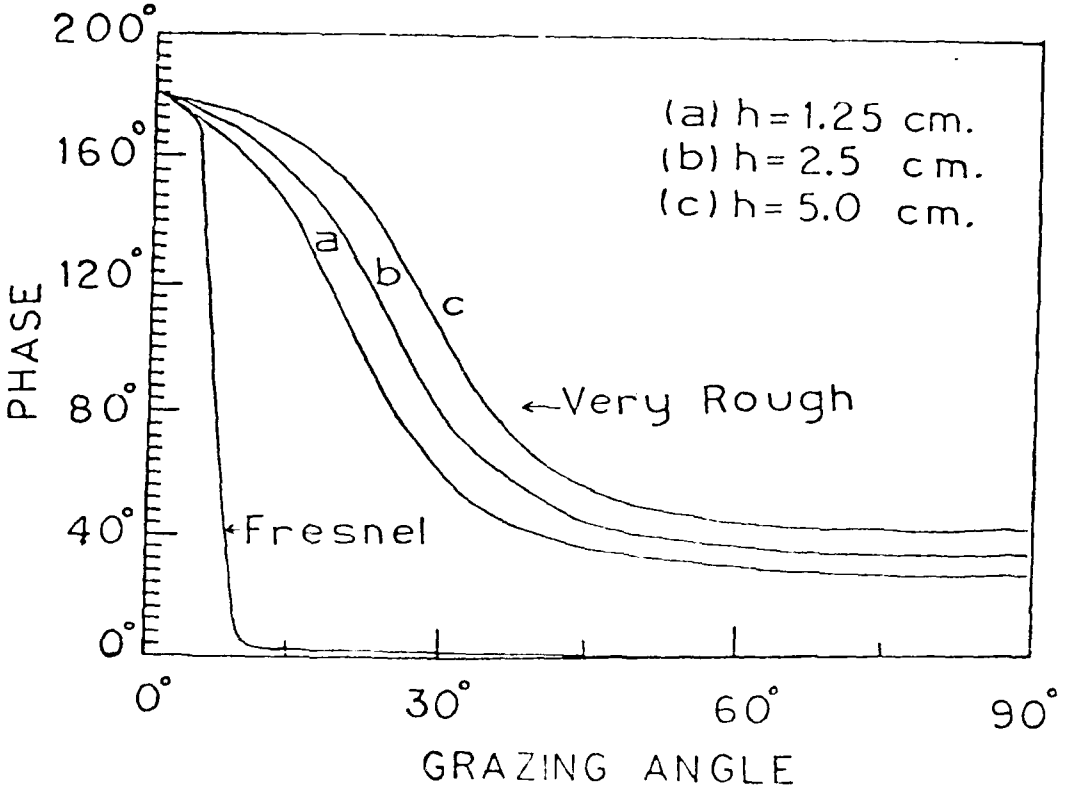

Figure 5.6: Phase of ρ_{VV} versus grazing angle and h.

At very low grazing angles (less than 5°) the magnitude and phases of the rough surface reflection coefficients differ little ($< 3\%$) from the corresponding quantities for a flat surface. However, the very rough surface scattering model is useful in representing regions of very rough geometry on a large-scale surface which otherwise satisfies the requirements of the Kirchhoff method. In this case the local angle of incidence at the very rough surface regions will assume all values in the range $0^\circ - 90^\circ$. Over this range of angles the reflection coefficients derived from the very rough surface scattering model depart significantly from the Fresnel reflection coefficient. Ignoring these differences can lead to significant signal processing errors.

5.6 Reflection Coefficients for a Gaussian Surface

For non-uniformly distributed surface profiles $f(z)$ is not linear; but we can construct a piecewise linear approximation to the actual $f(z)$ and then extend the methods of the preceding paragraphs to the resulting "multilayer" scattering problem. For example, suppose the surface profile $g(r,\omega)$ is Gaussian with zero mean and standard deviation s. We set

$$\frac{1}{2}h \triangleq 3s \tag{5.56}$$

which is based on the fact that the probability of a Gaussian random variable taking values outside $(-3s, 3s)$ is less than 1%.

Next we construct a piecewise linear approximation to $f(z)$ the Gaussian probability distribution function. Since $f(z)$ is an odd function of z, we need an odd number of

linear segments to approximate the profile. The approximation used is

$$f(z) \approx \begin{cases} m_2(z + h/2) & \text{if } -h/2 \le z \le -s \\ m_1(z + s) + m_2(h/2 - s) & \text{if } -s \le z \le s \\ m_2(z - s) + 2m_1 s + m_2(h/2 - s) & \text{if } s \le z \le h/2 \end{cases} \tag{5.57}$$

$$m_1 = \frac{1}{\sqrt{2\pi s}}, \quad m_2 = \frac{1 - \sqrt{2/pi}}{4s}$$

In this case we have approximated the layer in which surface variations occur (see Figure 5.2) by three layers each with a linear profile. Writing out the field expressions in terms of reflection and transmission coefficients and free parameters (A, B, C, etc.) and applying the boundary conditions (i) (ii) at the (four) artificial boundaries leads to a total of 16 equations for the coefficients of an arbitrarily polarized wave. For a wave with either horizontal or vertical polarization we have 8 (non-trivial) equations. In general, for an n–segment piecewise linear approximation, one must solve two sets of $(2n + 2)$ equations to compute ρ^q_{HH} and ρ^q_{VV}. The numerical implementation is straightforward.

In Figures 5.6 and 5.6 we show the reflection coefficients and for horizontally or vertically polarized radiation incident on a surface with a Gaussian profile approximated by three linear sections. The relevant parameters are $s = 0.2cm, h = 1.2cm, \lambda = 2.5cm, \epsilon_r = 80$, and $\sigma = 4.3mho/m$. We have also included plots of the Fresnel coefficients and the reflection coefficients for a linear profile distribution with $h = 1.2cm$. Note that the reflection coefficients are almost identical at low grazing angles where the scattering physics are insensitive to the surface profile. The coefficients differ significantly at larger grazing angles. This means that in situations like remote sensing of

the ocean surface or in high altitude radar engagements or in cases involving composite surface models where the (local) grazing angles could take values well away from 0, a careful description of the surface roughness distribution is important.

5.7 Applicability of the smoothing perturbation model to a Gaussian surface

Our purpose here is to provide a formal argument supporting use of the smoothing perturbation model for modeling random rough surfaces. We verify that the method applies to a Gaussian surface satisfying appropriate conditions. For simplicity, we consider a one–dimensional rough surface, $z = g(x, \omega)$.

Earlier we mentioned that the smoothing perturbation model applies to a random, very rough surface which is represented by a zero–mean, strict–sense stationary random process, defined on a probability triple (Ω, F, P), provided

$$\int_{-\infty}^{\infty} R(\tilde{x}, z, \tilde{z}) d\tilde{x} < \infty \tag{5.58}$$

where

$$R(\tilde{x}, z, \tilde{z}) = < \tilde{H}(x, z, \omega)\tilde{H}(x - \tilde{x}, \tilde{z}, \omega) > \tag{5.59}$$

$$\tilde{H}(x, z, \omega) = H(x, z, \omega) - H_0(z) \tag{5.60}$$

with $H(x, z, \omega)$ defined to be one for $z \geq g(x, \omega)$ and zero otherwise. In (5.60), $H_0(z)$ denotes the expected value of $H(x, z, \omega)$.

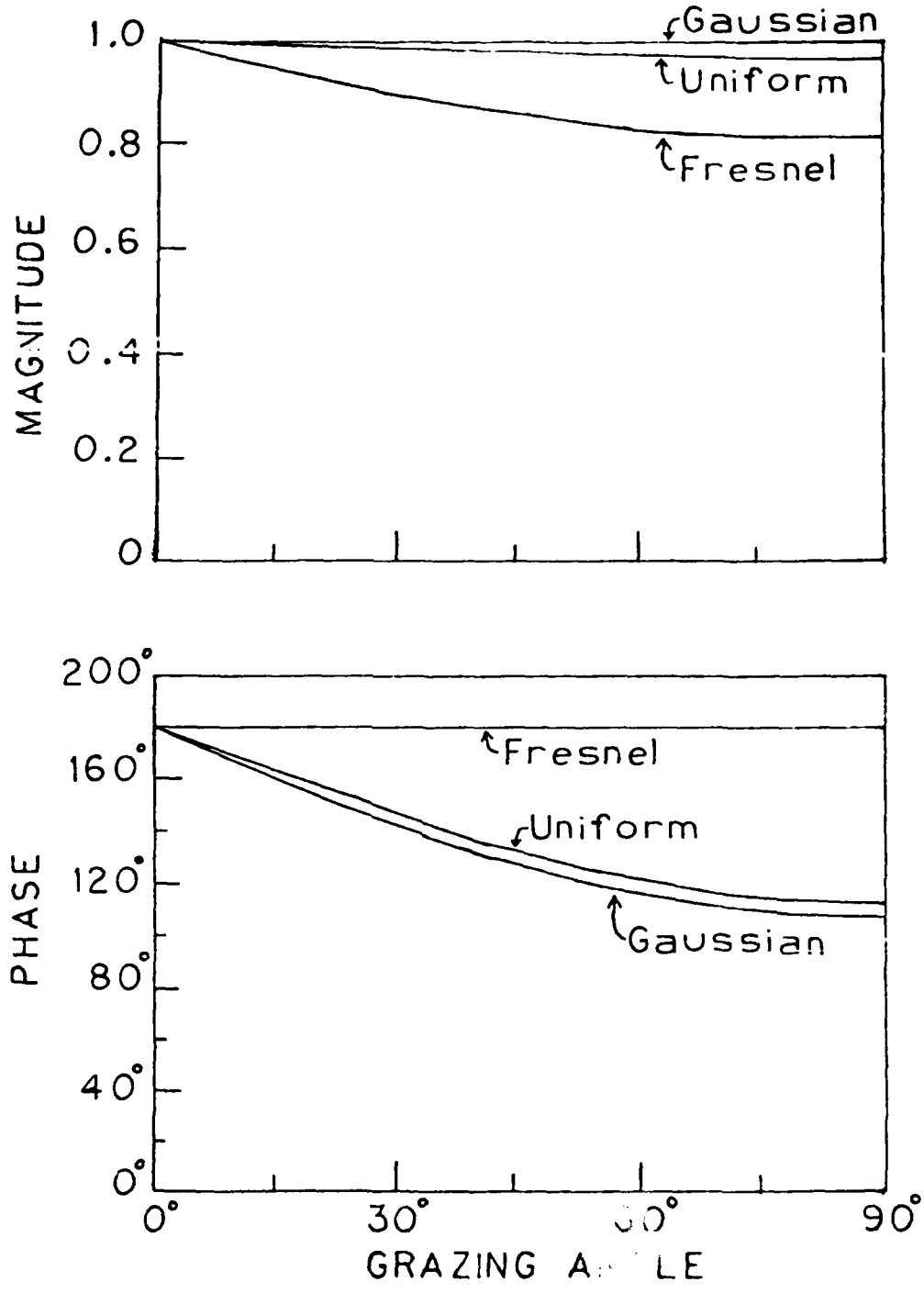

Figure 5.7: Magnitude and phase of ρ_{HH} for Gaussian surface.

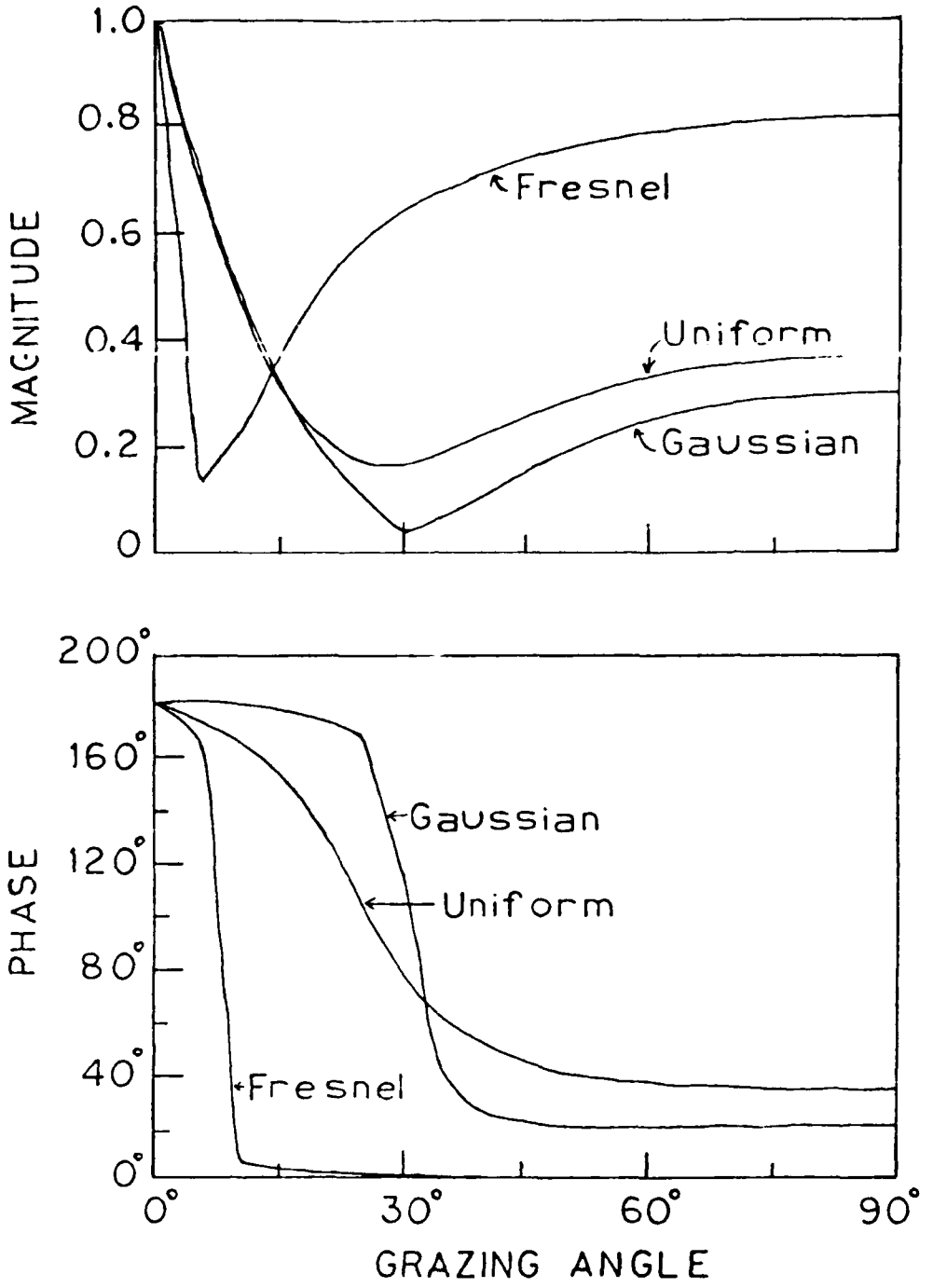

Figure 5.8: Magnitude and phase of ρ_{VV} for Gaussian surface.

From (5.59) we can write $R(\tilde{x}, z, \tilde{z})$ as

$$\begin{aligned} R(\tilde{x}, z, \tilde{z}) &= < H(x, z, \omega) H(x, \tilde{x}, \tilde{z}, \omega) > - H_0(z) H_0(\tilde{z}) \\ &= P\{g(x, \omega) \leq z, g(x - \tilde{x}, \omega) \leq \tilde{z})\} - H_0(z) H_0(\tilde{z}) \\ &= F_2(z, \tilde{z}, \tilde{x}) - H_0(z) H_0(\tilde{z}) \end{aligned} \tag{5.61}$$

where $F_2(\cdot, \cdot; x)$ denotes the joint distribution of $g(x, \omega)$ and $g(x - \tilde{x}, \omega)$. By definition, $H_0(\cdot)$ is the distribution function of $g(x, \omega)$.

Suppose the autocorrelation properties of the random process, $g(x, \omega)$, are such that the random variables $g(x, \omega)$ and $g(x - \tilde{x}, \omega)$ are *uncorrelated* for $x > d$ and for all $x \in \Re$. Then we have

$$F_2(z, \tilde{z}, \tilde{x}) = H_0(z) H_0(\tilde{z}), \quad |\tilde{x}| > d. \tag{5.62}$$

Using (5.61) and (5.62), we obtain

$$R(\tilde{x}, z, \tilde{z}) = 0, \quad |\tilde{x}| > d \tag{5.63}$$

in which case (5.58) is satisfied.

It is reasonable to expect that (5.58) will also be satisfied for surfaces, $g(x, \omega)$ with an autocorrelation function

$$R_g(\tilde{x}) = < g(x, \omega) g(x - \tilde{x}, \omega) > \tag{5.64}$$

which is finite for all $\tilde{x}$ and decays to zero as $|\tilde{x}| \to \infty$. Since $R(\tilde{x}, z, \tilde{z})$ given by (5.61) depends on the joint distribution function of the rough surface, we can provide only a formal argument supporting this contention.

When $g(x,\omega)$ is Gaussian, we can explicitly write the joint distribution function, $F_2(\cdot,\cdot;\tilde{x})$ of $g(x,\omega)$ and $g(x-\tilde{x},\omega)$ as

$$F_2(z,\tilde{z},;\tilde{x}) = \int_{-\infty}^{\tilde{z}}\int_{-\infty}^{z} \frac{e^{-\frac{s^2(u^2-2R_g(\tilde{x})uv/s^2+v^2}{2(s^4-R_g^2(\tilde{x}))}}}{2\pi\sqrt{s^4-R_g^2(\tilde{x})}}dudv \tag{5.65}$$

where $s = R_g(0)$ denotes the variance of the Gaussian surface roughness $g(x,\omega)$. Using (5.65), we can write (5.61) as

$$R(\tilde{x},z,\tilde{z}) = \int_{-\infty}^{\tilde{z}} \frac{e^{-v^2/2s^2}}{\sqrt{2\pi}s}\cdot\frac{1}{2}\left(1+erf\frac{s(z-R_g(\tilde{x})v/s^2)}{\sqrt{2(s^4-R_g^2(\tilde{x}))}}\right)dv$$

$$- H_0(z)H_0(\tilde{z}) \tag{5.66}$$

where

$$erfx = \int_{-\infty}^{x}\frac{2}{\sqrt{\pi}}e^{-t^2}dt \tag{5.67}$$

Now suppose, the autocorrelation function of the surface roughness, $R_g(\tilde{x})$, satisfies

$$R_g(\tilde{x}) < s^2e^{-\alpha|\tilde{x}|} \text{ for some } \alpha > 0. \tag{5.68}$$

Given a small $\delta > 0$, we can then find L such that for $|\tilde{x}| > L$

$$\frac{R_g(\tilde{x})}{s^2} < \delta e^{-\alpha|\tilde{x}-L|}. \tag{5.69}$$

Inspecting (5.66), we see that $\int_{-L}^{L} R(\tilde{x},z,\tilde{z}) = K(L)$ will be finite. Using (5.69), we find that for $|\tilde{x}| > L$, the argument of the error function in (5.66) is

$$\frac{s(z-R_g(\tilde{x}))v/s^2}{\sqrt{2(s^4-R_g^2(\tilde{x})}} = \frac{1}{\sqrt{2}s}\left(1-\delta^2e^{-2\alpha|\tilde{x}-L|}\frac{1}{2}(z-\delta ve^{-\alpha|\tilde{x}-L|})\right)$$

$$\approx \frac{z}{\sqrt{2}s} - \frac{\delta v}{\sqrt{2}s}e^{-\alpha|\tilde{x}-L|} + O(\delta^2)e^{-\alpha|\tilde{x}-L|}$$

so that by expanding, the error function, erf, in a Taylor series at $\frac{z}{\sqrt{2}s}$, we get

$$erf\frac{s(z-R_g(\tilde{x})v/s^2)}{\sqrt{2(s^4-R_g^2(\tilde{x})}} \approx erf\left(\frac{z}{\sqrt{2}s}\right) - \frac{2\delta v e^{-z^2/2s^2}e^{-\alpha|\tilde{x}-L|}}{\sqrt{2\pi}s} \tag{5.70}$$

$$+O(\delta^2)e^{-\alpha|\tilde{x}-L|}$$

Using (5.70) we can write (5.58) as

$$\int_{-\infty}^{\infty} R(\tilde{x},z,\tilde{z})dx = K(L) + \int_{|\tilde{x}|>L} d\tilde{x}[\int_{-\infty}^{\tilde{z}} \frac{e^{-v^2/2s^2}}{\sqrt{2\pi}s}dv\frac{1}{2}\left(1+erf\frac{z}{\sqrt{2}s}\right) \tag{5.71}$$

$$-H_0(z)H_0(\tilde{z}) - \frac{\delta e^{-z^2/2s^2}}{2\pi s^2}e^{-\alpha|\tilde{x}-L|}\int_{-\infty}^{\tilde{z}} ve^{-v^2/2s^2}dv + O(\delta^2)e^{-\alpha|\tilde{x}-L|}]$$

Using

$$H_0(z) = \int_{-\infty}^{z} \frac{e^{-u^2/2s^2}}{\sqrt{2\pi}s}du = \frac{1}{2}\left(1+erf\frac{z}{\sqrt{2}s}\right) \tag{5.72}$$

in (5.71), we get

$$\int_{-\infty}^{\infty} R(\tilde{x},z,\tilde{z}) = K(L) + O(\delta) < \infty \tag{5.73}$$

which shows that the smoothing perturbation model applies to Gaussian very rough surfaces with autocorrelation functions satisfying (5.68).

6 Conclusions

The purpose of this work was to develop analytical tools for the modeling and analysis of scattering of EM radiation in regions of foliage covered terrain. We have focussed on developing desciptions of the multiple scattering which occurs within a region containing a dense array of elementary scatterers. The three methods we have examined for this problem each have strong points: The T–matrix method leads naturally to approximations for the scattered field based on an explicit representation of the interaction of elementary scatterers (of several different classes). The computation of bounds based on the geometry of the underlying material provides tests for the evaluation of the approximate representations – e.g., approximations which provide estimates near the upper or lower bound on the effective parameter model of the region suggest that the model is valid. The homogenization technique, which is valid for low frequency radiation, provides not only a systematic method for the construction of approximations to the scattering physics; it also provides analytical methods for the evaluation of the precision of those estimates. That is, it is possible, at least in principle to estimate the quality of the approximation based on rigorous convergence arguments. Numerical evaluation of the models computed by homogenization for periodic media are straight forward. However, the case of random media is quite different and much more difficult.

We have also developed a computational procedure for computing the EM field scattered by a "very rough" random surface (above a region containing a dense array of elementary scatterers). In this scattering regime, the rms value h of the surface height variations is substantially larger than their correlation length ℓ, while λ, the

wavelength of the incident radiation is comparable to h. The method of smoothing perturbation is used to obtain a deterministic vector wave equation for the average electric field scattered by the random "very rough" surface. The coherent scattering is characterized by reflection coefficients which can be computed easily. The technique is applied to treat surfaces with uniform and Gaussian distributed height variations, and the results are compared to the Fresnel coefficients. Only small differences are observed at grazing incidence; however, at larger incident angles substantial differences (20-50%) in both amplitude and phase variations are obtained. Since the interaction of EM radiation with such surfaces involves local angles of incidence within the full range $(0 - 90°)$, errors can occur in approximations which ignore these differences.

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